(2.1) Functions F(x), G(x) and Related Integrals

$$\mathbf{F(x)=\int_0^x \frac{erf(w)}{w}dw,\quad G(x)=\int_x^\infty \frac{erfc(w)}{w}dw,\quad \int_0^x e^{-a^2w^2}\ln w\,dw,\quad \int_x^\infty e^{-a^2w^2}\ln w\,dw}$$

$$\boldsymbol{a > 0,\quad x > 0}$$

Alternate Formulas

(2.1.1) $$F(aX)=\int_0^X \frac{\operatorname{erf}(ax)}{x}dx\ ,\quad G(aX)=\int_X^\infty \frac{\operatorname{erfc}(ax)}{x}dx\,.$$

(2.1.2) $$\int_0^T e^{-a^2w^2}\ln w dw=\frac{\sqrt{\pi}}{2a}[\operatorname{erf}(aT)\ln T-F(aT)]$$

(2.1.3) $$\int_T^\infty e^{-a^2w^2}\ln w dw=\frac{\sqrt{\pi}}{2a}[\operatorname{erfc}(aT)\ln T+G(aT)].$$

(2.1.4) $$\int_0^\infty e^{-a^2w^2}\ln w dw=-\frac{\sqrt{\pi}}{2a}\left[\frac{\gamma}{2}+\ln(2a)\right].$$

Functional Relationships

(2.1.5) $$F(X)=\frac{\gamma}{2}+\ln(2X)+G(X)\qquad (\gamma=\text{Euler Constant}=0.5772156649015328606\ldots)$$

(2.1.6) $$\int_1^\infty e^{-ax}\frac{\ln x}{\sqrt{x}}dx=-\left[\frac{\partial}{\partial\nu}E_\nu(a)\right]_{\nu=1/2}=2\sqrt{\frac{\pi}{a}}G(\sqrt{a})=G_{1/2}(a)$$

where $G_\nu(a)$ is defined in Section **(2.2)** and explored fully in Chapter 3, Folder 18.

Convergent Series

(2.1.7) $$F(X)=\frac{2X}{\sqrt{\pi}}\sum_{k=0}^\infty \frac{(-1)^k X^{2k}}{k!(2k+1)^2}$$

(2.1.8) $$G(X)=\frac{1}{2}E_1(X^2)-\frac{1}{2\pi}\sum_{k=0}^\infty \frac{C_k}{(k+1/2)}E_{k+3/2}(X^2),\qquad C_k=\frac{(1/2)_k}{k!}$$

(2.1.9) $$G(X)=\frac{1}{2}E_1(X^2)-e^{-X^2}\ln 2+\frac{X^2}{2\pi}\sum_{k=0}^\infty C_k\frac{E_{k+1/2}(X^2)}{(k+1/2)^2}$$

Accelerated Series

(2.1.10) $$G(X)=\frac{1}{2}E_1(X^2)-\frac{1}{2\pi}H(X)$$

(2.1.11) $$H(X)=\sum_{k=0}^\infty \frac{C_k}{(k+1/2)}E_{k+3/2}(X^2)$$

$$=e^{-X^2}\sum_{k=1}^n W_k(-X^2)^{k-1}+\sqrt{\pi}\,ierfc(X)\sum_{k=1}^n S_{k,0}(-X^2)^{k-1}+(-X^2)^n\sum_{k=0}^\infty S_{n,k+1}E_{k+3/2}(X^2)$$

where

$$W_1 = \sum_{k=1}^{\infty} \frac{C_k}{(k+1/2)^2} = 2\pi \ln 2 - 4, \qquad W_n = \sum_{k=1}^{\infty} S_{n,k}, \qquad n \geq 1.$$

$$S_{1,k} = \frac{C_k}{(k+1/2)^2}, \quad k \geq 0. \qquad\qquad S_{n+1,k} = \frac{S_{n,k+1}}{(k+1/2)}, \quad k \geq 0, \quad n \geq 1.$$

Explicitly,

$$S_{10} = \frac{C_o}{(1/2)^2} = 4 \ , \ S_{n,k} = \frac{(1/2)_k}{(n+k-1)!(n+k-1/2)^2}, \qquad k \geq 0, \qquad n \geq 1$$

and numerical values for the W sequence are found in Chapter 3, Folder 16.

Asymptotic Series

(2.1.12) $$F(X) = \frac{2X}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k X^{2k}}{k!(2k+1)^2} \qquad \text{for } X \to 0$$

(2.1.13) $$G(X) = F(X) - \frac{\gamma}{2} - \ln(2X) \qquad \text{(See also } \mathbf{(2.1.9)}\text{)} \qquad \text{for } X \to 0$$

(2.1.14) $$\mathrm{G(X)} = \frac{1}{2X\sqrt{\pi}} \sum_{k=0}^{N} \frac{(-1)^k (1/2)_k}{X^{2k}} E_{k+3/2}(X^2) + W_N,$$

$$|W_N| \leq \frac{1}{2X\sqrt{\pi}} \frac{(1/2)_{N+1}}{X^{2N+2}} E_{N+5/2}(X^2) \qquad \text{for } X \to \infty$$

(2.1.15) $$F(X) = \frac{\gamma}{2} + \ln(2X) + G(X) \qquad \text{for } X \to \infty$$

LaPlace Transform

(2.1.16) $$\int_0^{\infty} e^{-pt} \frac{\ln t}{\sqrt{t}} dt = -\frac{\sqrt{\pi}}{\sqrt{p}}[\gamma + \ln(4p)], \qquad p > 0.$$

Computer Subroutines

F(X): DOUBLE PRECISION FUNCTION DFERF(…)
G(X): DOUBLE PRECISION FUNCTION DGERFC(…)

References : Chapter 3, Folders 6, 16 and 18

(2.2) Functions $E_\nu(x)$, $G_\nu(x)$ and Related Integrals

$$E_\nu(x) = \int_1^\infty \frac{e^{-xt}}{t^\nu} dt, \quad G_\nu(x) = \int_1^\infty \frac{E_\nu(xt)}{t^\nu} dt = -\frac{\partial E_\nu(x)}{\partial \nu} = \int_1^\infty \frac{e^{-xt}\ln t}{t^\nu} dt, \quad x > 0$$

For Integer and Half Odd Integer Orders with Application to

$$I_n(b,T) = \int_T^\infty \frac{e^{-b^2x^2}\ln x}{x^n} dx, \qquad n = 0, 1, 2, \ldots, \qquad T > 0$$

Introduction

The relation

(2.2.1) $\displaystyle\int_x^\infty \frac{E_\mu(t)}{t^\nu} dt = \frac{1}{x^{\nu-1}} \frac{E_\nu(x) - E_\mu(x)}{\mu - \nu}$, or equivalently $\displaystyle\int_1^\infty \frac{E_\mu(xt)}{t^\nu} dt = \frac{E_\nu(x) - E_\mu(x)}{\mu - \nu}$, $\mu \neq \nu$

can be verified by differentiation w.r.t. x. Notice that if $\mu \to \nu$, we get the case where $\mu = \nu$ on the left, and we define $G_\nu(x)$ by

(2.2.2) $\displaystyle G_\nu(x) \equiv \int_1^\infty \frac{E_\nu(xt)}{t^\nu} dt = \frac{-\partial E_\nu(x)}{\partial \nu} = \int_1^\infty \frac{e^{-xt}\ln t}{t^\nu} dt$ or $\displaystyle\int_x^\infty \frac{E_\nu(t)}{t^\nu} dt = \frac{1}{x^{\nu-1}} G_\nu(x)$, $\mu = \nu$

Basic Properties of $E_\nu(x)$ and $G_\nu(x)$, $\nu > 1$

Basic properties of the exponential integral

(2.2.3) $$E_\nu(x) = \int_1^\infty \frac{e^{-xt}}{t^\nu} dt, \qquad \nu \geq 0$$

can be found in most handbooks. In this section we repeat some of these properties and note the similarity of $E_\nu(x)$ and $G_\nu(x)$

(2.2.4) $E_{\nu+1}(x) < E_\nu(x)$

(2.2.5) $G_{\nu+1}(x) < G_\nu(x)$, $\quad \nu \geq 1$

(2.2.6) $$\frac{e^{-x}}{x+\nu} < E_\nu(x) \leq \frac{e^{-x}}{x+\nu-1}, \qquad \nu \geq 1$$

(2.2.7) $$\frac{e^{-x}}{(x+\nu)(x+\nu+1)} < \frac{E_{\nu+1}(x)}{x+\nu} < G_\nu(x) \leq \frac{E_\nu(x)}{x+\nu-1} \leq \frac{e^{-x}}{(x+\nu-1)^2}, \qquad \nu \geq 1$$

(2.2.8) $\nu E_{\nu+1}(x) + xE_\nu(x) = e^{-x}$

(2.2.9) $\nu G_{\nu+1}(x) + xG_\nu(x) = E_{\nu+1}(x)$, $\quad \nu \geq 1$

(2.2.10) $E'_\nu(x) = -E_{\nu-1}(x)$

(2.2.11) $G'_\nu(x) = -G_{\nu-1}(x)$, $\quad \nu > 1$

(2.2.12) $E_\nu(0) = 1/(\nu-1)$,

(2.2.13) $$G_\nu(0) = 1/(\nu-1)^2 \qquad \nu > 1$$

Closed Forms

(2.2.14) $$E_{1/2}(x) = \sqrt{\pi}\,\mathrm{erfc}\sqrt{x})/\sqrt{x}, \qquad E_{3/2}(x) = 2\sqrt{\pi}\,\mathrm{ierfc}(\sqrt{x})$$

(2.2.15) $$G_{1/2}(x) = 2\sqrt{\frac{\pi}{x}}G(\sqrt{x})$$

where G(x) is described in Section **(2.1)** and Chapter 3, Folder 16.

Convergent Series

(2.2.16) $$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-x)^n}{nn!}$$

(2.2.17) $$E_2(x) = x[\gamma + \ln x - 1] + 1 - \sum_{n=2}^{\infty} \frac{(-x)^n}{(n-1)n!}$$

(2.2.18) $$G_1(x) = \frac{\pi^2}{12} + \frac{1}{2}E_1^2(x) - (\gamma + \ln x)\sum_{n=1}^{\infty} \frac{(-x)^n}{nn!} + \sum_{n=1}^{\infty} \frac{(-x)^n}{n^2 n!} - \sum_{k=2}^{\infty} \frac{(-x)^k}{k} A_k$$

(2.2.19) $$G_2(x) = 1 - \frac{\pi^2}{12}x - \frac{x}{2}(\gamma + \ln x)^2 + x(\gamma + \ln x) - x - x\sum_{n=1}^{\infty} \frac{(-x)^n}{n^2(n+1)n!}$$

$$+ x\sum_{n=2}^{\infty} \frac{(-x)^n}{n(n+1)} A_n - \frac{x}{2}\sum_{n=2}^{\infty} \frac{(-x)^n}{n+1} D_n$$

where

$$A_n = \frac{1}{n!}\sum_{k=1}^{n-1} \frac{C_k^n}{n-k}, \qquad D_n = \frac{1}{n!}\sum_{k=1}^{n-1} \frac{C_k^n}{k(n-k)}, \qquad n \ge 2$$

and C_k^n are binomial coefficients. Two other options for $G_2(x)$ are also given in Chapter 3, Folder18.

Application of $E_\nu(x)$ and $G_\nu(x)$ to $\boldsymbol{I_n(b,T) = \int_T^\infty \frac{e^{-b^2x^2}\ln x}{x^n}dx,\ \ n = 0, 1, 2, \ldots,\ \ T > 0}$

Explicit formula

(2.2.20) $$I_n(b,T) = \frac{\ln T}{2T^{n-1}}E_{(n+1)/2}(b^2T^2) + \frac{1}{4T^{n-1}}G_{(n+1)/2}(b^2T^2)$$

For $n = 0$ the results are computed in **(2.1)**:

(2.2.21) $$I_0(b,T) = \int_T^\infty e^{-b^2x^2}\ln x\,dx = \frac{\sqrt{\pi}}{2b}[\mathrm{erfc}(bT)\ln T + G(bT)]$$

For even integers we get the G_ν function of half-odd orders:

(2.2.22) $$I_{2n}(b,T) = \frac{\ln T}{2T^{2n-1}} E_{n+1/2}(b^2T^2) + \frac{1}{4T^{2n-1}} G_{n+1/2}(b^2T^2) \qquad n \ge 0$$

For odd integers, we get the G_ν functions of integer orders,

(2.2.23) $$I_{2n+1}(b,T) = \frac{\ln T}{2T^{2n}} E_{n+1}(b^2T^2) + \frac{1}{4T^{2n}} G_{n+1}(b^2T^2) \qquad n \ge 0$$

Recurrence for $I_n(b,T)$:

(2.2.24) $$\frac{(n+1)}{2b^2} I_{n+2} + I_n = \frac{\ln T}{2b^2} \cdot \frac{e^{-b^2T^2}}{T^{n+1}} + \frac{1}{4b^2T^{n+1}} E_{(n+3)/2}(b^2T^2) \qquad n \ge 0$$

and this shows that the even and odd sequences are de-coupled. Also, the amplification factors (for the growth and/or decay of homogeneous solutions)

$$\frac{2b^2}{n+1} \text{ or } \frac{n+1}{2b^2}$$

for forward or backward recurrence show that a stable recurrence occurs (ratios less than or equal to 1) by recursion away from $N_b = [2b^2]$ or N_b+1 since one of these is an even index and one is odd. Notice that if T=1 then,

(2.2.25) $$I_n(b,1) = \frac{1}{4} G_{(n+1)/2}(b^2) = -\frac{1}{4}\left[\frac{\partial}{\partial \nu} \mathrm{E}_\nu(\mathrm{b}^2)\right]_{\nu=(n+1)/2}$$

Stability of Recurrence Relations

It is known that the recurrence for $E_\nu(x)$ is stable if one generates $E_\nu(x)$ for the order closest to x and recurs away from this index with the two-term recurrence relation above. This pattern also applies to $G_\nu(x)$ and this is what is done to generate sequences in the subroutines for the integer and half-odd integer orders.

Computer Subroutines

The computation of $E_\nu(x)$ and $G_\nu(x)$ is described in Chapter3, Folder18. These computations result in subroutines

$E_\nu(x)$: SUBROUTINE DEXINT(…) for sequences of positive integer orders
SUBROUTINE DHEXINT(…) for sequences of positive half-odd integer orders
$G_\nu(x)$: SUBROUTINE DGEXINT(…) for sequences of positive integer orders
SUBROUTINE DGHEXINT(…) for sequences of positive half-odd integer orders

References: Chapter 3, Folders 16 and 18

(2.3) Functions I(a,b,x), J(a,b,x), U(a,b,x), V(a,b,x) and Related Integrals

$$\mathbf{I(a,b,x)} = \int_x^\infty \mathbf{e^{-a^2w^2}\,erfc(bw)dw}, \quad \mathbf{J(a,b,x)} = \int_x^\infty \mathbf{e^{-a^2w^2}\,erf(bw)dw}$$

$$\mathbf{U(a,b,x)} = \int_0^x \mathbf{e^{-a^2w^2}\,erfc(bw)dw}, \quad \mathbf{V(a,b,x)} = \int_0^x \mathbf{e^{-a^2w^2}\,erf(bw)dw}$$

$$a \geq 0, \quad b \geq 0 \quad x>0$$

Other Notations

The I, J, U, V notation is commonly used to denote other integrals. Consequently, we often added a subscript 5 to these functions to distinguish them from other notations. The subscript 5 is a designation for Folder 5 in Chapter 3. Thus, I, J, U, and V can also be I_5, J_5, U_5, and V_5.

Convergent Series For I(a,b,x)

A series expansion is developed for

(2.3.1) $$I(a,b,x) = \frac{1}{2\sqrt{\pi}\sqrt{a^2+b^2}} G(a,b,x), \quad G(a,b,x) = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!}\left(\frac{a^2}{a^2+b^2}\right)^k E_{k+3/2}(d^2x^2) \quad a \leq b$$

(2.3.2) $$I(a,b,x) = \frac{be^{-a^2x^2}}{a^2+b^2}\sum_{k=1}^{\infty}\left(\frac{4a^2}{a^2+b^2}\right)^{k-1} [(k-1)!\, i^{2k-1}erfc(bx)], \qquad a \leq b$$

which converges for all a and b but the convergence is best for $a \leq b$. A companion relation

(2.3.3) $$I(a,b,x) = \frac{\sqrt{\pi}}{2a}\mathrm{erfc}(ax)\,\mathrm{erfc}(bx) - \frac{b}{a} I(b,a,x) \qquad a > b$$

is used for $a > b$ since $I(b,a,x)$ contains the factor $[b^2/(a^2+b^2)]^k$ and the convergence is best for $a > b$. $G(a,b,x)$ is further broken down into

(2.3.4) $$\frac{G(a,b,x)}{\sqrt{a^2+b^2}} = 2\frac{e^{-d^2x^2}}{a}\tan^{-1}\frac{a}{b} - dx^2 S(a,b,x), \qquad d^2 = a^2+b^2$$

$$S(a,b,x) = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!(k+1/2)}\left(\frac{a^2}{a^2+b^2}\right)^k E_{k+1/2}(d^2x^2), \qquad a \leq b$$

to break out the dominant behavior when x is small. Both of these series converge without the restrictions $a \leq b$ *or* $a > b$ but the restrictions ensure that the ratios

$$a^2/d^2 \quad or \quad b^2/d^2$$

do not exceed ½ and each series can be terminated in no more than 50 terms with errors $O(10^{-15})$. These formulae are manipulated in Chapter 3, Folder 5 to obtain forms which are suitable for computation.

Convergent Series For J(a,b,x)

Notice that

(2.3.5) $$J(a,b,x) = I(a,0,x) - I(a,b,x) = \frac{\sqrt{\pi}}{2a}\mathrm{erfc}(ax) - I(a,b,x)$$

The complete computation for numerical evaluation is given in Chapter 3, Folder 5.

Convergent Series For U(a,b,x)

Notice that

(2.3.6) $$U(a,b,x) = I(a,b,0) - I(a,b,x) = \frac{1}{a\sqrt{\pi}}\tan^{-1}\frac{a}{b} - I(a,b,x)$$

Further details of computation are given in Chapter 3, Folder 5.

Convergent Series For V(a,b,x)

Notice that

(2.3.7) $$V(a,b,x) = \frac{\sqrt{\pi}}{2a}\operatorname{erf}(ax) - U(a,b,x) = \frac{1}{a\sqrt{\pi}}\tan^{-1}\frac{b}{a} - J(a,b,x)$$

The details for a more robust numerical evaluation are given in Chapter 3, Folder 5.

Functional Relations

(2.3.8) $$aI(a,b,x) + bI(b,a,x) = \frac{\sqrt{\pi}}{2}\operatorname{erfc}(ax)\operatorname{erfc}(bx)$$

(2.3.9) $$aJ(a,b,x) + bJ(b,a,x) = \frac{\sqrt{\pi}}{2}\left[1 - \operatorname{erf}(ax)\operatorname{erf}(bx)\right] = \frac{\sqrt{\pi}}{2}\left[erfc(ax) + erf(ax)erfc(bx)\right]$$

(2.3.10) $$J(a,b,x) + I(a,b,x) = \frac{\sqrt{\pi}}{2a}\operatorname{erfc}(ax)$$

(2.3.11) $$aU(a,b,x) + bU(b,a,x) = \frac{\sqrt{\pi}}{2}\left[1 - \operatorname{erfc}(ax)\operatorname{erfc}(bx)\right] = \frac{\sqrt{\pi}}{2}\left[erf(ax) + erfc(ax)erf(bx)\right]$$

(2.3.12) $$a\,V(a,b,x) + b\,V(b,a,x) = \frac{\sqrt{\pi}}{2}erf(ax)erf(bx)$$

(2.3.13) $$U(a,b,x) + V(a,b,x) = \frac{\sqrt{\pi}}{2a}\operatorname{erf}(ax)$$

Asymptotics for $\mathbf{x} \to +\infty$

(2.3.14) $$I(a,b,x) = \frac{1}{2b\sqrt{\pi}}\sum_{k=0}^{N}\frac{(-1)^k(1/2)_k}{(bx)^{2k}}E_{k+1}\left[x^2(a^2+b^2)\right] + W_N(a,b,x)\ ,$$

$$\left|W_N(a,b,x)\right| \le \frac{1}{2b\sqrt{\pi}}\frac{(1/2)_{N+1}}{(bx)^{2N+2}}E_{N+2}\left[x^2(a^2+b^2)\right] \qquad a \le b$$

(2.3.15) $$I(a,b,x) = \frac{\sqrt{\pi}}{2a}erfc(ax)erfc(bx) - \frac{b}{2a^2\sqrt{\pi}}\sum_{k=0}^{N}\frac{(-1)^k(1/2)_k}{(ax)^{2k}}E_{k+1}\left[x^2(a^2+b^2)\right] + R_N(a,b,x)\ ,$$

$$\left|R_N(a,b,x)\right| \le \frac{b}{a}\left|W_N(b,a,x)\right| \qquad \text{a>b}$$

Now, we use relations

(2.3.16) $$\mathrm{J(a,b,}x) = \int_x^{\infty} \mathrm{e}^{-\mathrm{a}^2 w^2} \mathrm{erf(b}\,w)\,\mathrm{d}\,w = \frac{\sqrt{\pi}}{2a} erfc(ax) - I(a,b,x)\,,$$

(2.3.17) $$\mathrm{U(a,b,}x) = \int_0^{x} \mathrm{e}^{-\mathrm{a}^2 w^2} \mathrm{erfc(b}\,w)\,\mathrm{d}\,w, = I(a,b,0) - I(a,b,x) = \frac{1}{a\sqrt{\pi}} \tan^{-1}\frac{a}{b} - I(a.b.x)$$

(2.3.18) $$\mathrm{V(a,b,}x) = \int_0^{x} \mathrm{e}^{-\mathrm{a}^2 w^2} \mathrm{erf(b}\,w)\,\mathrm{d}\,w = \frac{\sqrt{\pi}}{2a} erf(ax) - U(a,b,x) = \frac{1}{a\sqrt{\pi}} \tan^{-1}\frac{b}{a} - J(a,b,x)$$

to get the expansions for J, U, and V.

Related Functions

A generalization of I(a,b,T)

Details relating to the manipulation of

(2.3.19) $$Y_n(a,b,T) = \int_T^{\infty} e^{-a^2w^2} i^n erfc(bw)dw, \qquad n \geq -1$$
$$a > 0, \quad b \geq 0$$

are found in Chapter 3, Folder 29 for the series representations:

For Case I, $a \leq b$,

(2.3.20) $$Y_n(a,b,T) = \frac{be^{-a^2T^2}}{a^2+b^2} \sum_{k=1}^{\infty} \left(\frac{4a^2}{a^2+b^2} \right)^{k-1} \frac{\Gamma(n/2+k)}{\Gamma(n/2+1)} i^{n+2k-1} erfc(bT), \quad \mathrm{n} \geq 0$$

and for case II, a>b,

(2.3.21) $$Y_n(a,b,T) = \frac{\sqrt{\pi}}{2a} \sum_{k=0}^{n} (-1)^k \left(\frac{b}{a} \right)^k i^k erfc(aT)\, i^{n-k} erfc(bT) + (-1)^{n+1} \left(\frac{b}{a} \right)^{n+1} Y_n(b,a,T)$$

with the special case

(2.3.22) $$Y_{-1}(a,b,T) = \frac{erfc(T\sqrt{a^2+b^2})}{\sqrt{a^2+b^2}}.$$

The recurrence

(2.3.23) $$Y_{n-2} = \frac{a^2}{a^2+b^2} \left[2nY_n + \frac{b}{a^2} e^{-a^2T^2} i^{n-1} erfc(bT) \right]$$

is numerically stable on backward recurrence and is used in subroutine INTEGI29 to generate even and odd sequences of Y_n.

Integral of I(a,b,T)

Integration of $Y_0(a,b,T) = I_5(a,b,T)$ for Case I, $a \leq b$, gives

(2.3.24) $$\int_T^{\infty} I_5(a,b,w)dw = \frac{b}{a^2+b^2} \sum_{k=1}^{\infty} \left(\frac{4a^2}{a^2+b^2} \right)^{k-1} [(k-1)!\, Y_{2k-1}(a,b,T)], \qquad a \leq b$$

and integration of Case II, $a > b$, in Chapter3, Folder 5 gives

(2.3.25) $$\int_T^{\infty} I_5(a,b,w)dw = \frac{\sqrt{\pi}}{2a} \int_T^{\infty} erfc(aw)erfc(bw)dw - \frac{b}{a} \int_T^{\infty} I_5(b,a,w)dw \qquad a > b$$

where the $I_5(b,a,T)$ integral on the right is computed above since the first parameter is less than the second parameter. The integral

$$\int_T^\infty erfc(aw)erfc(bw)dw$$

is computed in Chapter3, Folder 9, but we can get an alternate form from

$$Y_1(a,b,T) = \int_T^\infty e^{-a^2w^2}\,\mathrm{ierfc(bw)dw}$$

by integration by parts:

(2.3.26) $$\int_T^\infty erfc(aw)erfc(bw)dw = \frac{1}{b}\left[erfc(aT)\,ierfc(bT) - \frac{2a}{\sqrt{\pi}} Y_1(a,b,T\right] \ ,$$

with

(2.3.27) $$Y_1(a,b,T) = \frac{be^{-a^2T^2}}{a^2+b^2}\sum_{k=1}^{\infty}\left(\frac{4a^2}{a^2+b^2}\right)^{k-1}\frac{\Gamma(1/2+k)}{\Gamma(3/2)}i^{2k}erfc(bT), \quad \frac{\Gamma(1/2+k)}{\Gamma(3/2)} = 2\,(1/2)_k, \ \ a \le b$$

For this computation we can always choose $a \le b$ since the integral is symmetric in a and b and a can be chosen to be the smaller of the two parameters.

Representations of i^nerfc(z), n = -1,0,1

Take $b = 0$ in $I(a,b,x)$:

(2.3.28) $$\frac{\sqrt{\pi}}{2a}\mathrm{erfc}(ax) = \frac{1}{2\sqrt{\pi}}\frac{1}{a}\sum_{k=0}^{\infty} C_k E_{k+3/2}(a^2x^2)\,, \qquad C_k = (1/2)_k/k!$$

(2.3.29) $$\mathrm{erfc}(z) = \frac{1}{\pi}\sum_{k=0}^{\infty} C_k E_{k+3/2}(z^2)$$

Differentiate wrt z to get

(2.3.30) $$\frac{e^{-z^2}}{z} = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} C_k E_{k+1/2}(z^2)$$

Integrate erfc(z) above

(2.3.31) $$\mathrm{ierfc}(z) = \frac{1}{\sqrt{\pi}}\mathrm{erfc}(z) - \frac{z}{2\pi}\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!(k+1)}E_{k+3/2}(z^2)$$

LaPlace Transforms

(2.3.32) $$\int_0^\infty e^{-pt}\frac{\mathrm{erf}(\sqrt{\alpha t})}{\sqrt{t}}dt = \frac{2}{\sqrt{\pi p}}\tan^{-1}\sqrt{\frac{\alpha}{p}}$$

(2.3.33) $$\int_0^\infty e^{-pt}\sqrt{t}\,\mathrm{erf}(\sqrt{\alpha t})dt = \frac{1}{p\sqrt{\pi}}\left[\frac{1}{\sqrt{p}}\tan^{-1}\sqrt{\frac{\alpha}{p}} + \frac{\sqrt{\alpha}}{p+\alpha}\right]$$

(2.3.34) $$\int_0^\infty e^{-pt}\frac{\mathrm{erfc}(\sqrt{\alpha t})}{\sqrt{t}}dt = \frac{2}{\sqrt{\pi p}}\tan^{-1}\sqrt{\frac{p}{\alpha}}$$

(2.3.35) $$\int_0^\infty e^{-pt}\sqrt{t}\,\mathrm{erfc}(\sqrt{\alpha t})dt = \frac{1}{p\sqrt{\pi}}\left[\frac{1}{\sqrt{p}}\tan^{-1}\sqrt{\frac{p}{\alpha}} - \frac{\sqrt{\alpha}}{p+\alpha}\right]$$

Special Cases

(2.3.36) $$I(a,b,0)=\int_0^\infty e^{-a^2w^2}erfc(bw)dw=\frac{1}{a\sqrt{\pi}}\tan^{-1}\frac{a}{b}$$

(2.3.37) $$I(a,a,x)=\frac{\sqrt{\pi}}{4a}erfc^2(ax)$$

(2.3.38) $$J(a,b,0)=\int_0^\infty e^{-a^2w^2}\operatorname{erf}(bw)dw=\frac{1}{a\sqrt{\pi}}\tan^{-1}\frac{b}{a}$$

(2.3.39) $$V(0,b,x)=\int_0^x \operatorname{erf}(bw)dw=x\operatorname{erf}(bx)-\frac{1}{b\sqrt{\pi}}(1-e^{-b^2x^2})$$

Inequalities

A restatement of **(2.3.8)** gives the relation

(2.3.40) $$W\equiv\frac{\sqrt{\pi}}{2}\operatorname{erfc}(ax)\operatorname{erfc}(bx)=aI(a,b,x)+bI(b,a,x)$$

Divide W by $a+b$. Then the right side is a convex linear combination of $I(a,b,x)$ and $I(b,a,x)$. Therefore

(2.3.41) $$\min[I(a,b,x),I(b,a,x)]\le\frac{W}{a+b}\le\max[I(a,b,x),I(b,a,x)]$$

Divide W by $\sqrt{a^2+b^2}$ and we have the dot product of two vectors, one of which has length 1. By the Cauchy inequality, we have

(2.3.42) $$\left(\frac{W}{\sqrt{a^2+b^2}}\right)^2\le I^2(a,b,x)+I^2(b,a,x)$$

Notice that if $b=a$, then the max and min are the same and

(2.3.43) $$I(a,a,x)=\frac{W}{2a}=\frac{\sqrt{\pi}}{4a}\operatorname{erfc}^2(ax).$$

We have similar results for J(a,b,x):

(2.3.44) $$\overline{W}\equiv\frac{\sqrt{\pi}}{2}\left[1-\operatorname{erf}(ax)\operatorname{erf}(bx)\right]=aJ(a,b,x)+bJ(b,a,x)$$

(2.3.45) $$\min[J(a,b,x),\,J(b,a,x)]\le\frac{\overline{W}}{a+b}\le\max[J(a,b,x),\,J(b,a,x)]$$

(2.3.46) $$\left(\frac{\overline{W}}{\sqrt{a^2+b^2}}\right)^2\le J^2(a,b,x)+J^2(b,a,x)$$

If a = b then the max and min are equal and

(2.3.47) $$J(a,a,x)=\frac{\overline{W}}{2a}=\frac{\sqrt{\pi}}{4a}\left[1-\operatorname{erf}^2(ax)\right].$$

Because of **(2.3.11)** and **(2.3.12),** we also have similar relations for U and V.

Computer Subroutines

I(a,b,x): SUBROUTINE INTEGI5(…)
J(a,b,x): SUBROUTINE INTEGJ5(…)
V(a,b,x): SUBROUTINE INTEGV5(…)
Y_n (a,b,T) : SUBROUTINE INTEGI29(…)

References: Chapter 3, Folder 5, Chapter 3, Folder 29

(2.4) Functions $\mathbf{I_{20}(a,b,T)}$ and $\mathbf{I^c_{20}(a,b,T)}$

$$\mathbf{I_{20}(a,b,T) = \int_T^\infty e^{-a^2x^2}\,erf(bx)\ln x\,dx, \qquad I^c_{20}(a,b,T) = \int_T^\infty e^{-a^2x^2}\,erfc(bx)\ln x\,dx,}$$

$$\boldsymbol{a > 0,\quad b > 0,\quad T \geq 0}$$

Series Representations

Case I, $a \leq b$

(2.4.1)
$$\mathrm{I}^c_{20}(a,b,T) = \mathrm{I}_5(a,b,T)\ln T + \frac{E_1(X)}{2a\sqrt{\pi}}\tan^{-1}\left(\frac{a}{b}\right) - \frac{S(a,b,T)}{4\sqrt{\pi}\sqrt{a^2+b^2}} \qquad a \leq b$$

where $I_5(a,b,T)$ is the I function of Chapter 3, Folder 5 and

$$S(a,b,T) = \sum_{k=0}^{\infty} \frac{C_k}{(k+1/2)}\left(\frac{a^2}{a^2+b^2}\right)^k E_{k+3/2}(X), \quad X = T^2(a^2+b^2), \qquad C_k = (1/2)_k/k!, \qquad k \geq 0$$

Notice that $a^2/(a^2+b^2) \leq ½$ and the convergence of the S series is better than $O(2^{-k}k^{-3/2})$.

Case II, $a > b$ The reflexive relation

(2.4.2)
$$\mathrm{I}^c_{19}(a,b,T) = -\,\mathrm{erfc}(aT)\,\mathrm{erfc}(bT)\ln T + \frac{2a}{\sqrt{\pi}}\mathrm{I}^c_{20}(a,b,T) + \frac{2b}{\sqrt{\pi}}\mathrm{I}^c_{20}(b,a,T)$$

is derived in Chapter 3, Folder 20 and used to compute for $a > b$:

(2.4.3)
$$\mathrm{I}^c_{20}(a,b,T) = \frac{\sqrt{\pi}}{2a}[\mathrm{I}^c_{19}(a,b,T) + \mathrm{erfc}(aT)\mathrm{erfc}(bT)\ln T] - \frac{b}{a}\mathrm{I}^c_{20}(b,a,T) \qquad a > b$$
$$= I_5(a,b,T)\,\ln T + \frac{\sqrt{\pi}}{2a}I^c_{19}(a,b,T) - \frac{b}{a}\left[\frac{E_1(X)}{2b\sqrt{\pi}}\tan^{-1}\left(\frac{b}{a}\right) - \frac{S(b,a,T)}{4\sqrt{\pi}\sqrt{a^2+b^2}}\right]$$

where $\mathrm{I}^c_{20}(b,a,T)$ fits into Case I because the first parameter is smaller than the second parameter. $\mathrm{I}^c_{19}\mathbf{(a,b,T)}$ is computed in Chapter 3, Folder19. $\mathrm{I}^c_{20}\mathbf{(0,b,T)}$ is evaluated explicitly, and the results for $\mathrm{I}^c_{20}\mathbf{(a,0,T)}$ are given in Chapter 3, Folder 16. For I_{20}, we can use erfc(x)=1-erf(x) and get a pair from

(2.4.4)
$$I_{20}(a,b,T) = \frac{\sqrt{\pi}}{2a}[\mathrm{erfc}(aT)\ln T + G(aT)] - I^c_{20}(a,b,T)$$

and $I^c_{20}(a,b,T)$ above where G(x) is computed in Section **(2.1)**. However another analysis yields the pair

(2.4.5)
$$I_{20}(a,b,T) = J_5(a,b,T)\ln T + \frac{\sqrt{\pi}}{2a}G(aT) - \left[\frac{E_1(X)}{2a\sqrt{\pi}}\tan^{-1}\left(\frac{a}{b}\right) - \frac{S(a,b,T)}{4\sqrt{\pi}\sqrt{a^2+b^2}}\right], \qquad a \leq b$$

and for a>b, we have

(2.4.6)
$$I_{20}(a,b,T) = I_{20}(a,b,0) - \frac{\sqrt{\pi}}{2a}[\mathrm{erf}(aT)\mathrm{erf}(bT)\ln T - I_{19}(a,b,T)] + \frac{b}{a}\left[I_{20}(b,a,0) - I_{20}(b,a,T)\right].$$

Results for $T \to 0$ in $I_{20}(a,b,0)$ and $I_{20}(b,a,0)$ are given in Chapter 3, Folder 20.

Computer Subroutine

$\mathbf{I_{20}(a,b,T)}$ **and** $\mathbf{I^{c}_{20}(a,b,T)}$ **: SUBROUTINE INTEGI20(…)** on KODE=1 and KODE=2

References: Chapter 3, Folder 20

(2.5) Functions $\mathbf{P(a,b,T)}, \mathbf{P^c(a,b,T)}$ and $\mathbf{Q(a,b,T)}$

$$\mathbf{P(a,b,T)} = \int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erf}(bw)}{w}\,dw, \quad \mathbf{P^c(a,b,T)} = \int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erfc}(bw)}{w}\,dw,$$

$$\mathbf{Q(a,b,T)} = \int_T^\infty e^{-a^2w^2} E_1(b^2w^2)\,dw$$

$$\boldsymbol{a > 0, \quad b > 0, \quad T \geq 0}$$

Series Representations

(2.5.1)
$$P(a,b,T) = \begin{cases} \dfrac{1}{2}E_1(a^2T^2) - G(\sqrt{X}) - \ln\left[\dfrac{2\sqrt{a^2+b^2}}{b+\sqrt{a^2+b^2}}\right]\mathrm{erfc}(\sqrt{X}) + \dfrac{\sqrt{X}}{2\sqrt{\pi}}S_1(a,b,X), & a \leq b \\[2ex] \dfrac{1}{2}E_1(a^2T^2)\,\mathrm{erf}(bT) + \ln\left[\dfrac{b+\sqrt{a^2+b^2}}{a}\right]\mathrm{erfc}(\sqrt{X}) - \dfrac{bT}{\sqrt{\pi}}S_2(b,a,X), & a > b \end{cases}$$

(2.5.2)
$$Q(a,b,T) = \begin{cases} \dfrac{\sqrt{\pi}}{a}\ln\left[\dfrac{a+\sqrt{a^2+b^2}}{b}\right]\mathrm{erfc}(\sqrt{X}) - TS_2(a,b,X) & a \leq b \\[2ex] \dfrac{\sqrt{\pi}}{2a}E_1(b^2T^2)\,\mathrm{erfc}(aT) - \dfrac{\sqrt{\pi}}{a}\ln\left[\dfrac{2\sqrt{a^2+b^2}}{a+\sqrt{a^2+b^2}}\right]\mathrm{erfc}(\sqrt{X}) - \dfrac{\sqrt{\pi}}{a}G(\sqrt{X}) & \\ \qquad + \dfrac{\sqrt{X}}{2a}S_1(b,a,X) & a > b \end{cases}$$

$$S_1(a,b,X) = \sum_{k=1}^{\infty} C_k\left(\frac{a^2}{a^2+b^2}\right)^k \frac{E_{k+1/2}(X)}{k}, \qquad S_2(a,b,X) = \sum_{k=0}^{\infty}\left(\frac{a^2}{a^2+b^2}\right)^k \frac{E_{k+1}(X)}{2k+1}$$

$$X = T^2(a^2+b^2), \qquad C_k = \frac{(1/2)_k}{k!}, \quad \mathrm{k}=1,2,\ldots$$

and the computation of

(2.5.3)
$$G(x) = \int_x^\infty \frac{\mathrm{erfc}(v)}{v}\,dv$$

is described in **(2.1)**. Notice that whenever S_1 and S_2 are used, the powers of $a^2/(a^2+b^2)$ or $b^2/(a^2+b^2)$ are less than (or equal to) $1/2^k$, making the convergence of these series quite acceptable numerically.

Special Cases

(2.5.4)
$$P(a,b,0) = \ln\left[\frac{b+\sqrt{a^2+b^2}}{a}\right], \qquad Q(a,b,0) = \frac{\sqrt{\pi}}{a}\ln\left[\frac{a+\sqrt{a^2+b^2}}{b}\right]$$

$$P(a,0,T) = 0, \qquad Q(0,b,T) = \frac{\sqrt{\pi}}{b}\mathrm{erfc}(bT) - TE_1(b^2T^2)$$

Functional Relations

(2.5.5) $$P(a,b,T)+P^{c}(a,b,T)=\frac{1}{2}E_{1}(a^{2}T^{2}).$$

(2.5.6) $$P(a,b,T)=\frac{1}{2}E_{1}(a^{2}T^{2})\operatorname{erf}(bT)+\frac{b}{\sqrt{\pi}}Q(b,a,T).$$

To compute $P^{c}(a,b,T)$ from **(2.5.5)** and **(2.5.1)**, the subtractions should be done analytically to retain significant digits.

Asymptotics

For bT large, we have

(2.5.7) $$P(a,b,T)\sim\frac{1}{2}E_{1}(a^{2}T^{2})-\frac{1}{2bT\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(1/2)_{k}}{(bT)^{2k}}E_{k+3/2}(X),\qquad X=T^{2}(a^{2}+b^{2})$$

(2.5.8) $$P^{c}(a,b,T)\sim\frac{1}{2bT\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(1/2)_{k}}{(bT)^{2k}}E_{k+3/2}(X),\qquad X=T^{2}(a^{2}+b^{2})$$

(2.5.9) $$Q(a,b,T)\sim\frac{T}{2(bT)^{2}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(k)!}{(bT)^{2k}}E_{k+3/2}(X),\qquad X=T^{2}(a^{2}+b^{2})$$

For aT large, we use **(2.5.5)** in conjunction with **(2.5.8)** to compute P(a,b,T)

(2.5.10) $$P(a,b,T)=\frac{1}{2}E_{1}(a^{2}T^{2})\operatorname{erf}(bT)+\frac{b}{\sqrt{\pi}}Q(b,a,T)$$

(2.5.11) $$P(a,b,T)\sim\frac{1}{2}E_{1}(a^{2}T^{2})\operatorname{erf}(bT)+\frac{b}{\sqrt{\pi}}\left[\frac{T}{2(aT)^{2}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(k)!}{(aT)^{2k}}E_{k+3/2}(X)\right],\quad X=T^{2}(a^{2}+b^{2})$$

(2.5.12) $$P^{c}(a,b,T)\sim\frac{1}{2}E_{1}(a^{2}T^{2})\mathit{erfc}(bT)-\frac{b}{\sqrt{\pi}}\left[\frac{T}{2(aT)^{2}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(k)!}{(aT)^{2k}}E_{k+3/2}(X)\right],\ X=T^{2}(a^{2}+b^{2})$$

For large aT in Q(a,b,T), we solve for Q from **(2.5.5),** exchange parameters and use **(2.5.6),**

(2.5.13) $$Q(a,b,T)=\frac{\sqrt{\pi}}{a}\left[\ P(b,a,T)-\frac{1}{2}E_{1}(b^{2}T^{2})\operatorname{erf}(aT)\ \right]$$

(2.5.14) $$Q(a,b,T)\sim\frac{\sqrt{\pi}}{a}\left[\frac{1}{2}E_{1}(b^{2}T^{2})\ \operatorname{erfc}(\mathrm{aT})-\frac{1}{2aT\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^{k}\frac{(1/2)_{k}}{(aT)^{2k}}E_{k+3/2}(X)\right],\quad X=T^{2}(a^{2}+b^{2}).$$

For $a\to 0$ we get

(2.5.15) $$P(a,b,T)=\frac{1}{2}E_{1}(a^{2}T^{2})-G(bT)+O(a^{2})\ =\ -\frac{\gamma}{2}-\ln(aT)-G(bT)+O(a^{2})\qquad \mathrm{a}\to 0$$

where G(x) is defined in **(2.1)** with its properties displayed in Chapter 3, Folders 6 and 16.

Computer Subroutines

P(a,b,T) or $\mathbf{P^{c}(a,b,T)}$: SUBROUTINE INTEGP(…) with KODE=1 or KODE=2
Q(a,b,T): SUBROUTINE INTEGQ(…)

References: Chapter 3, Folders 6, 11, and 16

(2.6) Functions $I_1(a,b,T)$ and $I_1^c(a,b,T)$

$$I_1(a,b,T)=\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erf}(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{e^{-a^2/\tau}\,\mathrm{erf}(b/\sqrt{\tau})}{\sqrt{\tau}}d\tau$$

$$I_1^c(a,b,T)=\int_T^\infty \frac{e^{-a^2w^2}\,\mathrm{erfc}(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{e^{-a^2/\tau}\,\mathrm{erfc}(b/\sqrt{\tau})}{\sqrt{\tau}}d\tau$$

$$a>0,\quad b>0,\quad T=\frac{1}{\sqrt{t}},\quad t>0$$

Other Notations

See the Preface for a detailed explanation of the notation used in the edition dated August, 2003. In that version, I_1 and I_1^c were defined by the second integral of the title lines. In order to refer to the first integral, the symbols $\overline{I}_1$ or $\overline{I}_1^c$ were used. In this edition, I_1 and I_1^c are defined by the first integral of the title lines and there is no $\overline{I}_1$ or $\overline{I}_1^c$.

Representations

(2.6.1) $$I_1(a,b,T)=\frac{e^{-a^2T^2}\,\mathrm{erf}(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1(X)-2a^2J_5(a,b,T),\qquad X=(a^2+b^2)T^2$$

(2.6.2) $$\begin{aligned}I_1(a,b,T)&=\frac{e^{-a^2T^2}\,erf(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1(X)-a\sqrt{\pi}erfc(aT)+2a^2I_5(a,b,T)\\&=-\frac{e^{-a^2T^2}\,erfc(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1(X)+\frac{\sqrt{\pi}}{T}ierfc(aT)+2a^2I_5(a,b,T)\end{aligned}$$

(2.6.3) $$I_1(a,b,T)=\frac{\sqrt{\pi}}{T}\mathrm{erf}(bT)\mathrm{ierfc}(aT)+\frac{b}{\sqrt{\pi}}E_1(X)-2abI_5(b,a,T)$$

(2.6.4) $$I_1^c(a,b,T)=\frac{e^{-a^2T^2}\,\mathrm{erfc}(bT)}{T}-\frac{b}{\sqrt{\pi}}E_1(X)-2a^2I_5(a,b,T)$$

(2.6.5) $$I_1^c(a,b,T)=\frac{\sqrt{\pi}}{T}\mathrm{erfc}(bT)\mathrm{ierfc}(aT)-\frac{b}{\sqrt{\pi}}E_1(X)+2abI_5(b,a,T)$$

where J_5 and I_5 are the J and I integrals of Folder 5 in Chapter 3,

(2.6.6) $$\begin{aligned}J_5(a,b,T)&=J(a,b,T)=\int_T^\infty e^{-a^2x^2}\,\mathrm{erf}(bx)dx,\\I_5(a,b,T)&=I(a,b,T)=\int_T^\infty e^{-a^2x^2}\,\mathrm{erfc}(bx)dx\end{aligned}$$

Functional Relations

(2.6.7) $$I_1(a,b,T)+I_1^c(a,b,T)=\frac{\sqrt{\pi}}{T}\mathrm{ierfc}(aT)$$

Asymptotics for small t

(2.6.8) $$I_1^c(a,b,T) \sim \frac{b}{2\sqrt{\pi}} \sum_{k=0}^{N} C_k \left(\frac{t}{b^2}\right)^{k+1} E_{k+2}\left(\frac{a^2+b^2}{t}\right) + R_N, \qquad T = 1/\sqrt{t}$$

(2.6.9) $$I_1(a,b,T) = \sqrt{\pi t}\ \mathrm{ierfc}(a/\sqrt{t}) - I_1^c(a,b,T)$$

$$C_k = (-1)^k (1/2)_k, \qquad k = 0,1,\ldots \qquad |R_N| \le \frac{b}{2\sqrt{\pi}} |C_{N+1}| \left(\frac{t}{b^2}\right)^{N+2} E_{N+3}\left(\frac{a^2+b^2}{t}\right)$$

Special Cases,T>0

(2.6.10) b=0, $I_1(a,0,T) = 0$

(2.6.11) a=0, $$I_1(0,b,T) = \int_T^\infty \frac{\mathrm{erf}(bw)}{w^2} dw = \frac{\mathrm{erf}(bT)}{T} + \frac{b}{\sqrt{\pi}} E_1(b^2T^2)$$

Computer Subroutines

$\mathbf{I_1(a,b,T)}$ **or** $\mathbf{I_1^c(a,b,T)}$ **: SUBROUTINE INTEGI1(…)** with KODE=1 or KODE=2

References: Chapter 3, Folders 1, 2, and 10

(2.7) Functions $\mathbf{I_{13}(a,b,T)}$ and $\mathbf{I^c_{13}(a,b,T)}$

$$\mathbf{I_{13}(a,b,T)=\int_T^{\infty}\frac{e^{-a^2w^2}erf(bw)}{w^3}dw=\frac{1}{2}\int_0^t e^{-a^2/u}erf(b/\sqrt{u})du\,,}$$
$$\mathbf{I^c_{13}(a,b,T)=\int_T^{\infty}\frac{e^{-a^2w^2}erfc(bw)}{w^3}dw=\frac{1}{2}\int_0^t e^{-a^2/u}erfc(b/\sqrt{u})du}$$
$$\mathbf{a>0,\quad b>0,\quad \mathit{T}=\frac{1}{\sqrt{\mathit{t}}},\quad t>0}$$

Representations

(2.7.1) $$I_{13}(a,b,T)=\frac{e^{-a^2T^2}}{2T^2}\mathrm{erf}(bT)+\frac{b}{T}i\,\mathrm{erfc}(T\sqrt{a^2+b^2})-a^2P(a,b,T)$$

(2.7.2) $$I_{13}(a,b,T)=\frac{\mathrm{erf(bT)}}{2\mathrm{T}^2}\mathrm{E_2(a^2T^2)}+\frac{\mathrm{b}}{2\mathrm{T}\sqrt{\pi}}\mathrm{E_{3/2}[T^2(a^2+b^2)]}-\frac{\mathrm{a^2b}}{\sqrt{\pi}}\mathrm{Q(b,a,T)}$$

where P and Q are the functions of Chapter 3, Folder11. For the complementary integral, we have

(2.7.3) $$I^c_{13}(a,b,T)=\frac{e^{-a^2T^2}}{2T^2}erfc(bT)-\frac{a^2}{2}E_1(a^2T^2)-\frac{b}{T}ierfc(T\sqrt{a^2+b^2})+a^2P(a,b,T)$$

(2.7.4) $$\mathrm{I^c_{13}(a,b,T)=\frac{1}{2T^2}E_2(a^2T^2)erfc(bT)-\frac{b}{2T\sqrt{\pi}}E_{3/2}[T^2(a^2+b^2)]+\frac{a^2b}{\sqrt{\pi}}Q(b,a,T)}$$

(2.7.5) $$\mathrm{I^c_{13}(a,b,T)=\frac{1}{2T^2}E_2(a^2T^2)-I_{13}(a,b,T)}$$

where $E_{3/2}(x^2)=2\sqrt{\pi}\,ierfc(x)$ .Significant digits in **(2.7.3)** can saved if the E and P functions are combined analytically before evaluation (See **(2.5.1)**).

Computer Subroutine

$\mathbf{I_{13}(a,b,T)}$ **or** $\mathbf{I^c_{13}(a,b,T)}$**: SUBROUTINE INTEGI13(…)** with KODE=1 or KODE=2

Reference: Chapter 3, Folder13

(2.8) Functions $\mathbf{I_2(a,b,T)}$, $\mathbf{I_2^c(a,b,T)}$, $\mathbf{I_9(a,b,T)}$ and $\mathbf{I_9^c(a,b,T)}$

$$\mathbf{I_2(a,b,T)=\int_0^T erf(aw)\,erf(bw)dw=\frac{1}{2}\int_t^\infty \frac{erf(a/\sqrt{u})erf(b/\sqrt{u})}{u^{3/2}}du},$$

$$\mathbf{I_2^c(a,b,T)=\int_T^\infty erfc(aw)\,erfc(bw)dw=\frac{1}{2}\int_0^t \frac{erfc(a/\sqrt{u})erfc(b/\sqrt{u})}{u^{3/2}}du}$$

$$\mathbf{I_9(a,b,T)=\int_0^T w\,erf(aw)erf(bw)dw=\frac{1}{2}\int_t^\infty \frac{erf(a/\sqrt{u})erf(b/\sqrt{u})du}{u^2}}$$

$$\mathbf{I_9^c(a,b,T)=\int_T^\infty w\,erfc(aw)erfc(bw)dw=\frac{1}{2}\int_0^t \frac{erfc(a/\sqrt{u})erfc(b/\sqrt{u})du}{u^2}}$$

$$\mathbf{a>0,\quad b>0,\quad \boldsymbol{T}=\frac{1}{\sqrt{\boldsymbol{t}}},\quad t>0}$$

Representations

(2.8.1) $$I_2(a,b,T)=T\,\mathrm{erf}(aT)\,\mathrm{erf}(bT)+\frac{e^{-a^2T^2}}{a\sqrt{\pi}}\mathrm{erf}(bT)+\frac{e^{-b^2T^2}}{b\sqrt{\pi}}\mathrm{erf}(aT)-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\mathrm{erf}(T\sqrt{a^2+b^2})$$

Now from Chapter 3, Folder 7, we have

(2.8.2) $$\mathrm{I_2^c(a,b,T)=J_3(a,b,0,T)}$$
$$=\frac{\mathrm{e^{-a^2T^2}erfc(bT)}}{\mathrm{a}\sqrt{\pi}}+\frac{\mathrm{e^{-b^2T^2}erfc(aT)}}{\mathrm{b}\sqrt{\pi}}\mathrm{-Terfc(aT)erfc(bT)}-\frac{\sqrt{\mathrm{a^2+b^2}}}{\mathrm{ab}\sqrt{\pi}}\mathrm{erfc(T\sqrt{a^2+b^2})}$$

and the following expression resolves the indeterminant form for small a,

(2.8.3) $$J_3(a,b,0,T)=\mathrm{erfc}(aT)ierfc(bT)/b-S(a,b,0,T)$$

$$S(a,b,0,T)=\frac{a}{\pi b}\left[\frac{\sqrt{\pi}\,\mathrm{erfc}(TX)}{X+b}-\frac{bT}{2X}\sum_{m=0}^{\infty}\frac{(3/2)_m}{(m+1)!}\left(\frac{a^2}{a^2+b^2}\right)^m E_{m+3/2}(T^2X^2)\right]$$
$$X^2=a^2+b^2,\qquad a\le b.$$

Since $\mathrm{I_2^c(a,b,T)}$ is symmetric in a and b, we can always take a to be the smaller of the two parameters giving rapid convergence in the series since $a^2/(a^2+b^2)\le 1/2$ and the other terms are $O(1/\sqrt{m})$.

(2.8.4) $$I_2^c(a,b,0)=J_3(a,b,0,0)=\frac{1}{a\sqrt{\pi}}+\frac{1}{b\sqrt{\pi}}-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}=\frac{(2/\sqrt{\pi})}{a+b+\sqrt{a^2+b^2}}$$

The power series for $I_2(a,b,T)$ is

(2.8.5) $$I_2(a,b,T)=\int_0^T erf(aw)erf(bw)dw$$
$$=\frac{4abT^3}{\pi}\sum_{k=0}^{\infty}\frac{U_k(a^2,b^2)T^{2k}}{2k+3}=\frac{4T(aT)(bT)}{\pi}\sum_{k=0}^{\infty}\frac{U_k(a^2T^2,b^2T^2)}{2k+3}$$

where

(2.8.6) $$U_k(a^2,b^2)=\sum_{m=0}^{k} C_m(a^2)C_{k-m}(b^2), \qquad C_k(a^2)=\frac{(-1)^k a^{2k}}{k!(2k+1)}, \quad k\ge 0$$

and

(2.8.7) $I_2^c(a,b,T)=$

$$\frac{(2/\sqrt{\pi})}{a+b+\sqrt{a^2+b^2}} - T + \frac{2T(aT)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{C_k(a^2T^2)}{2k+2} + \frac{2T(bT)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{C_k(b^2T^2)}{2k+2} - \frac{4T(aT)(bT)}{\pi}\sum_{k=0}^{\infty}\frac{U_k(a^2T^2,b^2T^2)}{2k+3}.$$

A computational form for $I_2^c(a,b,T)$ for aT and bT bounded away from zero (larger parameters) is

(2.8.8) $$I_2^c(a,b,T)=\frac{T}{(aT)(bT)}\left[\frac{ierfc(T\sqrt{a^2+b^2})}{\sqrt{\pi}} - ierfc(aT)ierfc(bT)\right]$$

From Chapter3, Folder 29, we also have

(2.8.9) $$I_2^c(a,b,T)=\int_T^{\infty} erfc(aw)erfc(bw)dw=\frac{1}{b}\left[erfc(aT)\,ierfc(bT)-\frac{2a}{\sqrt{\pi}}Y_1(a,b,T\right] ,$$

with

(2.8.10) $$Y_1(a,b,T)=\frac{be^{-a^2T^2}}{a^2+b^2}\sum_{k=1}^{\infty}\left(\frac{4a^2}{a^2+b^2}\right)^{k-1}\frac{\Gamma(1/2+k)}{\Gamma(3/2)}i^{2k}erfc(bT), \quad \frac{\Gamma(1/2+k)}{\Gamma(3/2)}=2\,(1/2)_k, \quad a\le b$$

For this computation we can always choose $a\le b$ since the integral is symmetric in a and b and a can be chosen to be the smaller of the two parameters. This makes the convergence of $Y_1(a,b,T)$ rapid since $a^2/(a^2+b^2)\le 1/2$ and the other factors are $O(1/\sqrt{k})$.

Other representations which resolve the indeterminant forms **(2.8.1)** and **(2.8.2)** for a or b to zero are

(2.8.11) $$\begin{aligned}I_2(a,b,T)&=Terf(aT)erf(bT)+\frac{e^{-b^2T^2}}{b\sqrt{\pi}}erf(aT)-2\left(\frac{1}{a\sqrt{\pi}}+R\right)e^{-a^2T^2/2}\sinh(a^2T^2/2)\\&-Re^{-a^2T^2}erf(bT)+RT\sqrt{a^2+b^2}\,e^{-a^2T^2}\sum_{n=1}^{\infty}(-2)^n i^n erfc(bT)(bT\phi)^{n-1}\end{aligned}$$

where

(2.8.12) $$R=\frac{1}{b\sqrt{\pi}}\cdot\frac{a}{b+\sqrt{a^2+b^2}}, \qquad bT\phi=\frac{a^2T}{b+\sqrt{a^2+b^2}}$$

and, without loss of generality we take $a\le b$. For $I_2^c(a,b,T)$ and $a\le b$, we have

(2.8.13) $$\begin{aligned}I_2^c(a,b,T)&=erfc(aT)\frac{ierfc(bT)}{b}-Re^{-a^2T^2}erfc(bT)\\&-RTe^{-a^2T^2}\sqrt{a^2+b^2}\sum_{n=1}^{\infty}(-2)^n i^n erfc(bT)(bT\phi)^{n-1}\end{aligned}$$

The powers in the series are less than one if (aT) is less than $\left(1+\sqrt{2}\right)$ with $a\le b$ and the series converges rapidly since

$$i^n erfc(bT)\le i^n erfc(0)=\frac{1}{2^n\Gamma(n/2+1)}, \qquad n\ge 0.$$

For $I_9(a,b,T)$ and $I_9^c(a,b,T)$, we get

(2.8.14)
$$2I_9(a,b,T)=T\cdot I_2(a,b,T)-\frac{1}{a\sqrt{\pi}}V_5(a,b,T)-\frac{1}{b\sqrt{\pi}}V_5(b,a,T)$$
$$-\frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\left[T\,erf(T\sqrt{a^2+b^2})-\frac{1}{\sqrt{\pi}\sqrt{a^2+b^2}}\left(1-e^{-T^2(a^2+b^2)}\right)\right]$$

(2.8.15)
$$2I_9(a,b,T)=T^2erf(aT)erf(bT)+\frac{Te^{-a^2T^2}erf(bT)}{a\sqrt{\pi}}+\frac{Te^{-b^2T^2}erf(aT)}{b\sqrt{\pi}}$$
$$-\frac{1}{a\sqrt{\pi}}V_5(a,b,T)-\frac{1}{b\sqrt{\pi}}V_5(b,a,T)-\frac{1}{\mathrm{ab}\pi}\left(1-e^{-T^2(a^2+b^2)}\right)$$

(2.8.16)
$$2I_9^c(a,b,T)=T\cdot I_2^c(a,b,T)+\frac{1}{a\sqrt{\pi}}I_5(a,b,T)+\frac{1}{b\sqrt{\pi}}I_5(b,a,T)-\frac{1}{ab\sqrt{\pi}}ierfc(T\sqrt{a^2+b^2})$$

and

(2.8.17)
$$2I_9^c(a,b,T)=-\mathrm{T}^2\mathrm{erfc(aT)erfc(bT)}+\frac{\mathrm{Te}^{-a^2T^2}\mathrm{erfc(bT)}}{\mathrm{a}\sqrt{\pi}}+\frac{\mathrm{Te}^{-b^2T^2}\mathrm{erfc(aT)}}{\mathrm{b}\sqrt{\pi}}+\frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{I}_5\mathrm{(a,b,T)}+\frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{I}_5\mathrm{(b,a,T)}$$
$$-\frac{1}{\mathrm{ab}\sqrt{\pi}}\frac{\mathrm{e}^{-T^2(a^2+b^2)}}{\sqrt{\pi}}=\frac{-ierfc(aT)ierfc(bT)}{ab}+\frac{1}{a\sqrt{\pi}}I_5(a,b,T)+\frac{1}{b\sqrt{\pi}}I_5(b,a,T)$$

The V_5 and I_5 functions are the V and I functions of Chapter3, Folder 5.

The corresponding expressions for $I_9(a,b,T)$ and $I_9^c(a,b,T)$ for $a\le b$ which remove the indeterminancies for a to zero are

(2.8.18)
$$2I_9(a,b,T)=T^2erf(aT)erf(bT)+\frac{Te^{-b^2T^2}erf(aT)}{b\sqrt{\pi}}-\frac{1}{b\sqrt{\pi}}V_5(b,a,T)+U(a,b,T)$$
$$U(a,b,T)=H_1(a,b,T)+H_2(a,b,T)-W(a,b,T)\ ,\qquad \mathrm{a}\le\mathrm{b}$$

and

(2.8.19)
$$2I_9^c(a,b,T)\ =\ \frac{\mathrm{Terfc(aT)}}{\mathrm{b}}ierfc(bT)+\frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{I}_5\mathrm{(b,a,T)}+\mathrm{W(a,b,T)},\qquad \mathrm{a}\le\mathrm{b}$$

where

$$H_1(a,b,T)=\frac{1}{a\sqrt{\pi}}\left[Te^{-a^2T^2}-\frac{\sqrt{\pi}}{2a}erf(aT)\right],\qquad H_2(a,b,T)=\left[\frac{1}{a^2\pi}\tan^{-1}\left(\frac{a}{b}\right)-\frac{1}{ab\pi}\right]$$
$$\mathrm{W(a,b,T)}=-\frac{\mathrm{e}^{-a^2T^2}\mathrm{ierfc(bT)}}{\mathrm{ab}\sqrt{\pi}}+\frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{I}_5\mathrm{(a,b,T)}=\frac{1}{\mathrm{a}\sqrt{\pi}}\left[-\frac{\mathrm{e}^{-a^2T^2}\mathrm{ierfc(bT)}}{\mathrm{b}}+\frac{\mathrm{ierfc(T\sqrt{a^2+b^2})}}{\mathrm{d}}\right]+\mathrm{G}_9\mathrm{(a,b,T)}$$
$$G_9(a,b,T)=\frac{1}{2\pi d^2}\sum_{k=1}^{\infty}\frac{(1/2)_k}{k!}\left(\frac{a}{d}\right)^{2k-1}E_{k+3/2}\left(d^2T^2\right),\qquad d^2=a^2+b^2.$$

and the representations which remove the indeterminancies for a to zero are

$$H_1(a,b,T)=\frac{2\mathrm{T}^2}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^k(aT)^{2k-1}}{(k-1)!(2k+1)}\ ,\qquad H_2(a,b,T)=\frac{1}{b^2\pi}\sum_{k=1}^{\infty}\frac{(-1)^k(a/b)^{2k-1}}{(2k+1)},\quad \frac{\mathrm{a}}{\mathrm{b}}<1$$
$$\mathrm{W(a,b,T)}=\frac{\mathrm{(a/b)e}^{-a^2T^2}}{\sqrt{\pi}\sqrt{\mathrm{a}^2+\mathrm{b}^2}(b+\sqrt{\mathrm{a}^2+\mathrm{b}^2})}\left[-\mathrm{ierfc(bT)}+\mathrm{bT}\sum_{n=2}^{\infty}(-2)^{\mathrm{n-1}}\mathrm{n\ i^n erfc(bT)(bT}\phi)^{\mathrm{n}-2}\right]+\mathrm{G}_9(a,b,T)$$

Asymptotics for small and large T

For $I_2^c(a,b,T)$, **(2.8.4)** gives the case for $T \to 0$. For $I_2(a,b,T)$, **(2.8.1)** gives

(2.8.20) $$I_2(a,b,T) \sim T - \frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}} \text{ for } T \to \infty.$$

For $I_9^c(a,b,T)$ we have

(2.8.21) $$I_9^c(a,b,0) = \frac{1}{2ab\pi}\left[\frac{b}{a}\tan^{-1}\left(\frac{a}{b}\right) + \frac{a}{b}\tan^{-1}\left(\frac{b}{a}\right) - 1\right].$$

For $I_9(a,b,T)$ and $T \to \infty$,

(2.8.22) $$I_9(a,b,T) \sim \frac{1}{2}\left[T^2 - \frac{1}{a^2\pi}\tan^{-1}\left(\frac{b}{a}\right) - \frac{1}{b^2\pi}\tan^{-1}\left(\frac{a}{b}\right) - \frac{1}{ab\pi}\right], \qquad \mathrm{T} \to \infty$$

follows from **(2.8.15)** with

$$V_5(a,b,T) \sim \frac{1}{a\sqrt{\pi}}\tan^{-1}\left(\frac{b}{a}\right), \qquad \mathrm{T} \to \infty.$$

Computer Subroutines

$\mathbf{I_2(a,b,T)}$ or $\mathbf{I_2^c(a,b,T)}$: **SUBROUTINE INTEGI2(…)** with KODE=1 or KODE=2
$\mathbf{I_9(a,b,T)}$ or $\mathbf{I_9^c(a,b,T)}$: **SUBROUTINE INTEGI9(…)** with KODE=1 or KODE=2

References: Chapter 3, Folders 7, 9, and 29

(2.9) Function $\mathbf{I_{19}(a,b,T)}$ and $\mathbf{I^c_{19}(a,b,T)}$

$$\mathbf{I_{19}(a,b,T)=\int_0^T\frac{erf(aw)erf(bw)}{w}dw=\frac{1}{2}\int_t^\infty\frac{erf(a/\sqrt{\tau})erf(b/\sqrt{\tau})}{\tau}d\tau}$$
$$\mathbf{I^c_{19}(a,b,T)=\int_T^\infty\frac{erfc(aw)erfc(bw)}{w}dw=\frac{1}{2}\int_0^t\frac{erfc(a/\sqrt{\tau})erfc(b/\sqrt{\tau})}{\tau}d\tau}$$
$$\boldsymbol{a>0,\quad b>0,\quad T=1/\sqrt{t},\quad t>0}$$

Representations

(2.9.1)
$$\mathrm{I^c_{19}(a,b,T)=G(bT)erfc(aT)-\frac{a}{2\pi\sqrt{a^2+b^2}}\sum_{k=0}^{\infty}C_k\left(\frac{a^2}{a^2+b^2}\right)^k G_{k+3/2}(X)}$$
$$+\frac{a}{\sqrt{\pi}}\ln\left[\frac{b^2}{a^2+b^2}\right]I_5(a,b,T) \qquad a\le b$$
$$+\frac{a}{2\pi\sqrt{a^2+b^2}}\sum_{k=1}^{\infty}\left(\frac{a^2}{a^2+b^2}\right)^k E_{k+3/2}(X)\sum_{m=1}^{k}\frac{C_{k-m}}{m},\qquad X=T^2(a^2+b^2).$$

where $G(X)$ is the function of Chapter 3, Folder 16; $I_5(a,b,T)$ is the I function of Chapter 3, Folder 5, $C_k=(1/2)_k/k!$, $k\ge 0$, and $G_{k+3/2}(X)$ and $E_{k+3/2}(X)$ are the functions of Chapter 3, Folder18. The convergence of the series is rapid since $a^2/(a^2+b^2)$ is at most ½.

The representation for $a>b$ is obtained by exchanging a and b since $\mathrm{I^c_{19}(a,b,T)}$ is symmetric in a and b. For $\mathrm{I_{19}(a,b,T)}$ we get

(2.9.2) $\mathrm{I_{19}(a,b,T)=F(aT)-[G(bT)-I^c_{19}(a,b,T)]_{T\to 0}+[G(bT)-I^c_{19}(a,b,T)]},\qquad a\le b$

(2.9.3) $\mathrm{I_{19}(a,b,T)=F(bT)-[G(aT)-I^c_{19}(b,a,T)]_{T\to 0}+[G(aT)-I^c_{19}(b,a,T)]},\qquad a>b$

and the limit for $T\to 0$ is computable from $\mathrm{I^c_{19}}$ above:

(2.9.4) $G(bT)-I^c_{19}(a,b,T)=G(bT)-G(bT)erfc(aT)-R(a,b,T)=G(bT)erf(aT)-R(a,b,T)$

where R is the regular part of $\mathrm{I^c_{19}}$,

(2.9.5)
$$\mathrm{R(a,b,T)=\frac{a}{\sqrt{\pi}}\ln\left[\frac{b^2}{a^2+b^2}\right]I_5(a,b,T)-\frac{a}{2\pi\sqrt{a^2+b^2}}\sum_{k=0}^{\infty}C_k\left(\frac{a^2}{a^2+b^2}\right)^k G_{k+3/2}(X)}$$
$$a\le b$$
$$+\frac{a}{2\pi\sqrt{a^2+b^2}}\sum_{n=1}^{\infty}\left(\frac{a^2}{a^2+b^2}\right)^n E_{n+3/2}(X)\sum_{m=1}^{n}\frac{C_{n-m}}{m},\qquad X=T^2(a^2+b^2)$$

with $C_k=(1/2)_k/k!$, $G_{k+3/2}(0)=\dfrac{1}{(k+1/2)^2}$, $\quad E_{n+3/2}(0)=\dfrac{1}{(n+1/2)}$, $\quad I_5(a,b,0)=\dfrac{1}{a\sqrt{\pi}}\tan^{-1}\dfrac{a}{b}$

Then the difference is

(2.9.6) $$\left[\ G(bT) - I_{19}^{c}(a,b,T)\ \right]_{T\to 0} = -R(a,b,0), \qquad a \le b$$

since from Chapter 3, Folder18,

(2.9.7) $$G(bT)erf(aT) = erf(aT)\left[\ F(bT) - \frac{\gamma}{2} - \ln(2bT)\ \right] \to 0 \quad as \quad T \to 0 .$$

For a>b we simply exchange a and b in the formulas since both I_{19} and I_{19}^{c} are symmetric in a and b.

Computer Subroutine

$\mathbf{I_{19}(a,b,T)}$ or $\mathbf{I_{19}^{c}(a,b,T)}$ **: SUBROUTINE INTEGI19(…)** with KODE=1 or KODE=2

References: Chapter 3, Folders 5, 16, 18 and 19

(2.10) Functions $I_6(a,b,T)$ and $I_6^c(a,b,T)$ and Related Integrals

$$\mathbf{I_6(a,b,T)=\int_T^\infty \frac{erf(aw)erf(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{1}{\sqrt{u}}erf\left(\frac{a}{\sqrt{u}}\right)erf\left(\frac{b}{\sqrt{u}}\right)du}\,,$$

$$\mathbf{I_6^c(a,b,T)=\int_T^\infty \frac{erfc(aw)erfc(bw)}{w^2}dw=\frac{1}{2}\int_0^t \frac{1}{\sqrt{u}}erfc\left(\frac{a}{\sqrt{u}}\right)erfc\left(\frac{b}{\sqrt{u}}\right)du}$$

$$\mathbf{a>0,\quad b>0,\quad \mathit{T}=\frac{\mathit{1}}{\sqrt{\mathit{t}}},\quad t>0}$$

Other Notations

The symbol I(a,b,T) is a common symbol for integrals and is used in Folder 6 where the development occurs. We chose to define this integral for this part of the presentation as $I_6(a,b,T)$ to make it unique.

Representations

Symmetric Form

(2.10.1)

$$\begin{aligned}I_6(a,b,T) &= \frac{\mathrm{erf}(aT)\mathrm{erf}(bT)}{T}+\frac{aE_1(a^2T^2)}{\sqrt{\pi}}\mathrm{erf}(bT)\\ &+\frac{bE_1(b^2T^2)}{\sqrt{\pi}}\mathrm{erf}(aT)+\frac{2}{\sqrt{\pi}}\left[a\ln\left(\frac{b+\sqrt{a^2+b^2}}{a}\right)+b\ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)\right]\mathrm{erfc}(\sqrt{X})\\ &-\frac{2abT}{\pi}\left[\sum_{k=0}^{\infty}\left(\frac{a^2}{a^2+b^2}\right)^k\frac{E_{k+1}(X)}{2k+1}+\sum_{k=0}^{\infty}\left(\frac{b^2}{a^2+b^2}\right)^k\frac{E_{k+1}(X)}{2k+1}\right],\end{aligned}$$

$T=1/\sqrt{t}$, $X=(a^2+b^2)/t=T^2(a^2+b^2)$.

Non-symmetric form for $a\le b$ is

(2.10.2)

$$\begin{aligned}I_6(a,b,T) &= \mathrm{erf}(aT)\left[\frac{\mathrm{erf}(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1(b^2T^2)\right]+\frac{a}{\sqrt{\pi}}E_1(a^2T^2)\\ &-\frac{2a}{\sqrt{\pi}}G(\sqrt{X})+\frac{2}{\sqrt{\pi}}\left[-a\ln\left(\frac{2\sqrt{a^2+b^2}}{b+\sqrt{a^2+b^2}}\right)+b\ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)\right]\mathrm{erfc}(\sqrt{X})\\ &-\frac{2abT}{\pi}E_1(X)+\frac{a\sqrt{X}}{\pi}\sum_{k=1}^{\infty}\frac{(1/2)_k}{k!}\left(\frac{a^2}{a^2+b^2}\right)^k\frac{E_{k+1/2}(X)}{k}\\ &-\frac{2abT}{\pi}\sum_{k=1}^{\infty}\left(\frac{a^2}{a^2+b^2}\right)^k\frac{E_{k+1}(X)}{2k+1},\end{aligned}$$

$T=1/\sqrt{t}$, $X=(a^2+b^2)/t=T^2(a^2+b^2)$

Non-symmetric form for $a\ge b$

Since $I_6(a,b,T)$ is symmetric in a and b, we exchange a and b above for the case $a\ge b$.

The symmetric form is computationally efficient for *a* and *b* relatively close together. When *a* and *b* are widely separated, one of the series in $a^2/(a^2+b^2)$ or $b^2/(a^2+b^2)$ can be slowly convergent. Therefore the non-symmetric form was developed for overall computational use. In the non-symmetric forms, the series in $a^2/(a^2+b^2)$ or $b^2/(a^2+b^2)$ are rapidly convergent since each ratio is at most ½. The G function is addressed in Chapter 3, Folders 6 and 16.

From Chapter 3, Folder 15, we have $I_6(a,b,T)$ in terms of the basic functions P and Q of Folder 11,

(2.10.3) $$I_6(a,b,T)=\frac{\operatorname{erf}(aT)\operatorname{erf}(bT)}{T}+\frac{2a}{\sqrt{\pi}}P(a,b,T)+\frac{2b}{\sqrt{\pi}}P(b,a,T)$$

(2.10.4) $$I_6(a,b,T)=\operatorname{erf}(aT)\left[\frac{\operatorname{erf}(bT)}{T}+\frac{b}{\sqrt{\pi}}E_1(b^2T^2)\right]+\frac{2a}{\sqrt{\pi}}P(a,b,T)+\frac{2ab}{\pi}Q(a,b,T)$$

(2.10.5) $$P(a,b,T)+P^c(a,b,T)=\frac{1}{2}E_1(a^2T^2)$$

(2.10.6) $$P(a,b,T)=\frac{1}{2}E_1(a^2T^2)\operatorname{erf}(bT)+\frac{b}{\sqrt{\pi}}Q(b,a,T)$$

(2.10.7) $$I_6(a,b,T)=\frac{erf(aT)erf(bT)}{T}+\frac{1}{\sqrt{\pi}}\left[aE_1(a^2T^2)erf(bT)+bE_1(b^2T^2)erf(aT)\right]$$
$$+\frac{2ab}{\pi}[Q(a,b,T)+Q(b,a,T)]$$

(2.10.8) $$I_6^c(a,b,T)=\frac{\mathrm{erfc(aT)erfc(bT)}}{\mathrm{T}}-\frac{2\mathrm{a}}{\sqrt{\pi}}\mathrm{P^c(a,b,T)}-\frac{2\mathrm{b}}{\sqrt{\pi}}\mathrm{P^c(b,a,T)}$$

(2.10.9) $$I_6^c(a,b,T)=\mathrm{erfc(aT)}\left[\frac{\mathrm{erfc(bT)}}{\mathrm{T}}-\frac{\mathrm{b}}{\sqrt{\pi}}\mathrm{E_1(b^2T^2)}\right]-\frac{2\mathrm{a}}{\sqrt{\pi}}\mathrm{P^c(a,b,T)}+\frac{2\mathrm{ab}}{\pi}\mathrm{Q(a,b,T)}$$

(2.10.10) $$I_6^c(a,b,T)=\frac{erfc(aT)erfc(bT)}{T}-\frac{1}{\sqrt{\pi}}\left[aE_1(a^2T^2)erfc(bT)+bE_1(b^2T^2)erfc(aT)\right]$$
$$+\frac{2ab}{\pi}[Q(a,b,T)+Q(b,a,T)]$$

Special Cases

(2.10.11) $$\mathrm{I_6(a,b,0)}=\int_0^\infty\frac{\mathrm{erf(aw)erf(bw)}}{\mathrm{w}^2}\mathrm{dw}=\frac{2}{\sqrt{\pi}}\left[\mathrm{a}\ln\left(\frac{\mathrm{b}+\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{a}}\right)+\mathrm{b}\ln\left(\frac{\mathrm{a}+\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{b}}\right)\right]$$

Quadrature

In Chapter 3, Folder 3, a special quadrature procedure was developed. to overcome the slow convergence of the tail of I(a,b,T). This involves using the approximation erf(x)=1 + $O(10^{-16)})$ for $x \ge 6$.The procedure is as follows:

Let $W_m = \min\left(\frac{6}{a},\frac{6}{b}\right)$, $W_M = \max\left(\frac{6}{a},\frac{6}{b}\right)$, $X=\min(a,b)$, $T=1/\sqrt{t}$.

Then,

Case I: $T \le W_m$, $$I=\int_T^{W_m}\frac{\operatorname{erf}(aw)\operatorname{erf}(bw)}{w^2}dw+P(X,W_m,W_M)+\frac{1}{W_M}$$

Case II: $W_m < T \le W_M$, $$I = P(X,T,W_M)+\frac{1}{W_M}$$

Case III: $W_M < T < \infty$, $$I = 1/T = \sqrt{t}$$

where $$\frac{1}{W_M} + P(X,L,W_M) = \frac{erf(XL)}{L} + \frac{erfc(XW_M)}{W_M} + \frac{X}{\sqrt{\pi}}\left[E_1(X^2L^2) - E_1(X^2W_M^2)\right]$$

and the integral on $T \leq W_m$ is the quadrature part. For computation, the SLATEC library code DGAUS8 is a suitable, high accuracy quadrature code.

Computer Subroutines

$\mathbf{I_6(a,b,T)}$ **or** $\mathbf{I_6^c(a,b,T)}$: **SUBROUTINE INTEGI6(…)** with KODE=1 or K0DE=2
SUBROUTINE I6QUAD(…)

References: Chapter 3, Folders 3, 6, 11, 15

(2.11) Functions $W_3(a,b,T)$ and $W_3^c(a,b,T)$ and Related Integrals

$$\mathbf{W_3(a,b,T) = \int_T^\infty \frac{erf(aw)\,erf(bw)}{w^3}dw = \frac{1}{2}\int_0^t erf(a/\sqrt{\tau})\,erf(b/\sqrt{\tau})\,d\tau}$$

$$\mathbf{W_3^c(a,b,T) = \int_T^\infty \frac{erfc(aw)\,erfc(bw)}{w^3}dw = \frac{1}{2}\int_0^t erfc(a/\sqrt{\tau})\,erfc(b/\sqrt{\tau})\,d\tau}$$

$$\mathbf{J_\nu(a,T) = \int_T^\infty \frac{erf(aw)}{w^\nu}dw\,,\ \nu > 1\,, \qquad J_\nu^c(a,T) = \int_T^\infty \frac{erfc(aw)}{w^\nu}dw\,,\ \nu \neq 1}$$

$$\mathbf{a > 0, \quad b > 0, \quad T = \frac{1}{\sqrt{t}}, \quad t > 0}$$

Explicit Representations

(2.11.1) $$W_3(a,b,T) = \frac{\mathrm{erf}(aT)\,\mathrm{erf}(bT)}{2T^2} + \frac{1}{\sqrt{\pi}}[aI_1(a,b,T) + bI_1(b,a,T)]$$

(2.11.2) $$W_3^c(a,b,T) = \frac{\mathrm{erfc}(aT)\mathrm{erfc}(bT)}{2T^2} - \frac{1}{\sqrt{\pi}}[aI_1^c(a,b,T) + bI_1^c(b,a,T)]$$

(2.11.3) $$J_\nu(a,T) = \int_T^\infty \frac{\mathrm{erf}(aw)}{w^\nu}dw = \frac{1}{(\nu-1)T^{\nu-1}}\left[\mathrm{erf}(aT) + \frac{aT}{\sqrt{\pi}}E_{\nu/2}(a^2T^2)\right], \qquad \nu > 1$$

(2.11.4) $$J_\nu^c(a,T) = \int_T^\infty \frac{\mathrm{erfc}(aw)}{w^\nu}dw = \frac{1}{(\nu-1)T^{\nu-1}}\left[\mathrm{erfc}(aT) - \frac{aT}{\sqrt{\pi}}E_{\nu/2}(a^2T^2)\right], \qquad \nu \neq 1$$

(2.11.5) $$J_2(a,T) = \int_T^\infty \frac{\mathrm{erf}(aw)}{w^2}dw = \frac{\mathrm{erf}(aT)}{T} + \frac{a}{\sqrt{\pi}}E_1(a^2T^2), \qquad \nu = 2$$

(2.11.6) $$J_2^c(a,T) = \int_T^\infty \frac{\mathrm{erfc}(aw)}{w^2}dw = \frac{\mathrm{erfc}(aT)}{T} - \frac{a}{\sqrt{\pi}}E_1(a^2T^2), \qquad \nu = 2$$

(2.11.7) $$J_3(a,T) = \frac{\mathrm{erf}(aT)}{2T^2} + \frac{a}{T}\mathrm{ierfc}(aT) \qquad \nu = 3$$

(2.11.8) $$J_3^c(a,T) = \frac{\mathrm{erfc}(aT)}{2T^2} - \frac{a}{T}\mathrm{ierfc}(aT) = \frac{2i^2 erfc(aT)}{T^2} \qquad \nu = 3$$

Functional Relations

(2.11.9) $$W_3^c(a,b,T) - W_3(a,b,T) = \frac{1}{2T^2} - J_3(a,T) - J_3(b,T) = J_3^c(a,T) - J_3(b,T)$$

(2.11.10) $$J_3(a,T) + J_3^c(a,T) = \frac{1}{2T^2}$$

(2.11.11) $$J_\nu(a,T) + J_\nu^c(a,T) = \frac{1}{(\nu-1)T^{\nu-1}} \qquad \nu > 1.$$

Computer Subroutines

$W_3(a,b,T)$ or $W_3^c(a,b,T)$: SUBROUTINE INTEGW3(…) with KODE=1 or KODE=2

References Chapter 3, Folder 10

(2.12) Functions $\mathbf{I_3(a,b,c,T)}$ and $\mathbf{I_3^c(a,b,c,T) = J_3(a,b,c,T)}$

$$\mathbf{I_3(a,b,c,T) = \int_T^\infty e^{-c^2w^2} erf(aw)\, erf(bw)dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^{3/2}} erf\left(\frac{a}{\sqrt{u}}\right) erf\left(\frac{b}{\sqrt{u}}\right) du}\,,$$

$$\mathbf{I_3^c(a,b,c,T) = J_3(a,b,c,T) = \int_T^\infty e^{-c^2w^2} erfc(aw)\, erfc(bw)dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^{3/2}} erfc\left(\frac{a}{\sqrt{u}}\right) erfc\left(\frac{b}{\sqrt{u}}\right) du}$$

$$\mathbf{a \ge 0,\ b \ge 0,\ c>0,\ T = \frac{1}{\sqrt{t}},\quad t>0}$$

Series Representations

(2.12.1) $$I_3(a,b,c,T) = \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(cT) - I_5(c,a,T) - I_5(c,b,T) + J_3(a,b,c,T)$$

(2.12.2) $$J_3(a,b,c,T) = \int_T^\infty e^{-c^2w^2}\operatorname{erfc}(aw)\operatorname{erfc}(bw)dw$$

CASE I ($a,c \le b$) ($c \le a \le b$ or $a \le c \le b$)

(2.12.3) $$J_3(a,b,c,T) = \operatorname{erfc}(aT)I_5(c,b,T) - \frac{2a}{\sqrt{\pi}}\int_T^\infty e^{-a^2w^2} I_5(c,b,w)dw = \operatorname{erfc}(aT)I_5(c,b,T) - S(a,b,c,T)$$

$$S(a,b,c,T) = \frac{a}{\pi d}\left[\frac{\sqrt{\pi}\operatorname{erfc}(TX)}{X+d}\sum_{k=0}^{\infty}\frac{(1/2)_k}{(k+1)!}\left(\frac{c^2}{d^2}\right)^k P_k(y)\right.$$
$$\left. - \frac{(TX)d}{2X^2}\sum_{m=0}^{\infty}\frac{(3/2)_m}{(m+1)!}E_{m+3/2}(T^2X^2)\sum_{k=0}^{m}\binom{m}{k}\frac{(a^2/X^2)^{m-k}(c^2/X^2)^k}{2k+1}\right]$$

$d^2(b,c) = b^2+c^2$, $\quad X^2 = a^2+b^2+c^2$, $\quad y = a^2/(X+d)^2$, $\quad P_o(y) = 1$

$P_k(y) = F(-k, 1, k+2, y)$, $\quad k \ge 1$ $\quad a,c \le b$.

Here F(-k, 1, k+2, y) is the Gauss hypergeometric function, which for this application, can be computed with the series

(2.12.4) $$F(a,b,c,z) = \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!\,(c)_m}z^m, \qquad (a)_0 = 1,\quad (a)_m = (a)(a+1)...(a+m-1),\quad m \ge 1$$

With the first parameter a negative integer(-k), this F function is a polynomial of degree k since $(-k)_m = (-k)(-k+1)...(-k+m-1)$ is zero for m=k+1 and higher indices. I_5 is the I function of Chapter 3, Folder5.

An alternate formulation of this case is presented in Chapter 3, Folder 28 where S(a,b,c,T) is computed by the series

(2.12.5) $$S(a,b,c,T) = \frac{2a}{\sqrt{\pi}}\int_T^\infty e^{-a^2w^2} I_5(c,b,w)dw = \frac{a}{\pi d}\sum_{k=0}^{\infty} C_k\left(\frac{c^2}{d^2}\right)^k R_k(a,d,T),$$

where

$$R_k(a,d,T) = \int_T^\infty e^{-a^2w^2}E_{k+3/2}(d^2w^2)dw, \qquad k \ge 0,$$

$$d^2 = b^2 + c^2 \qquad\qquad C_k = (1/2)_k/k!.$$

Here $R_k(a,d,T)$ is expanded in the form

$$R_k(a,d,T) = \sqrt{\pi}\sum_{m=1}^{k+1} A(k+1,m)Y_{2m-1}(a,d,T)$$

where $Y_{2m-1}(a,d,T)$ is computed by backward recurrence on the two term relation

$$Y_{n-2} = \frac{a^2}{a^2+d^2}\left[2nY_n + \frac{d}{a^2}e^{-a^2T^2}i^{n-1}erfc(dT)\right]$$

starting at some large *odd* index N. The details of computation as well as the coefficients A(k+1,m) are given in Chapter 3, Folder 28. Another more desirable form for S is presented in Chapter 3, Folder 29:

(2.12.6) $$S(a,b,c,T) = \frac{2ab}{\sqrt{\pi}\sqrt{b^2+c^2}}\sum_{k=1}^{\infty}\left(\frac{4c^2}{b^2+c^2}\right)^{k-1}[(k-1)!Y_{2k-1}(\sqrt{a^2+c^2},b,T)]\,, \qquad a\ ,\ c \le b$$

where sequences of the integral

$$Y_n(a,b,T) = \int_T^{\infty} e^{-a^2w^2} i^n erfc(bw)dw, \qquad a>0, \quad b \ge 0, \quad n \ge -1$$

are computed by a series in conjunction with the recurrence above in subroutine INTEGI29.

CASE II ($a \le b \le c$)

(2.12.7) $$J_3(a,b,c,T) = \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(aT)\operatorname{erfc}(bT)\operatorname{erfc}(cT) - \frac{a}{c}J_3(b,c,a,T) - \frac{b}{c}J_3(a,c,b,T)$$

Notice that *c* is larger than *a* and *b*. This is precisely the criteria to compute $J_3(b,c,a,T)$ and $J_3(a,c,b,T)$ by Case I. This relation is analogous to

(2.12.8) $$I_5(a,b,T) = \frac{\sqrt{\pi}}{2a}\operatorname{erfc}(aT)\operatorname{erfc}(bT) - \frac{b}{a}I_5(b,a,T)$$

in Chapter 3, Folder 5 when $a \ge b$.

Power Series for small aT, bT, cT

(2.12.9) $$I_3(a,b,c,T) = I_3(a,b,c,0) - \int_0^T e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw$$

And using the power series for each erf function, we get

Case I, *a, c* $\le$ *b*

(2.12.10) $$I_3(a,b,c,0) = \frac{1}{c\sqrt{\pi}}\tan^{-1}\frac{a}{c} - S(a,b,c,0)$$

$$S(a,b,c,0) = \frac{a}{d\sqrt{\pi}}\cdot\frac{1}{X+d}\sum_{k=0}^{\infty}\frac{(1/2)_k}{(k+1)!}\left(\frac{c^2}{d^2}\right)^k P_k(y)$$

where $d^2 = b^2+c^2$, $X^2 = a^2+b^2+c^2$, $y = a^2/(X+d)^2$, $P_0(y) = 1$, $P_k(y) = F(-k, 1, k+2, y)$, $k \ge 1$ for *a, c* $\le$ *b* and $a = \min(a,b)$, $b = \max(a,b)$. In Case II, *c* takes the place of *b* as the dominant parameter.

Case II, $a \le b \le c$

(2.12.11) $$I_3(a,b,c,0) = \frac{a}{c} S(b,c,a,0) + \frac{b}{c} S(a,c,b,0)$$

Now the power series for the integral is

(2.12.12) $$\int_0^T e^{-c^2w^2} \operatorname{erf}(aw)\operatorname{erf}(bw)dw = \frac{4abT^3}{\pi} \sum_{k=0}^{\infty} \frac{V_k T^{2k}}{2k+3}$$

$$V_k T^{2k} = \sum_{n=0}^{k} D_n(c^2T^2) U_{k-n}(a^2T^2, b^2T^2), \quad U_k(a^2T^2, b^2T^2) = \sum_{m=0}^{k} C_m(a^2T^2) C_{k-m}(b^2T^2)$$

$$C_k(x^2) = \frac{(-1)^k x^{2k}}{k!(2k+1)}, \qquad D_k(x^2) = \frac{(-1)^k x^{2k}}{k!}$$

Series Expansion for Large c

(2.12.13)
$$\int_T^{\infty} e^{-c^2w^2} \operatorname{erf}(aw)\operatorname{erf}(bw)dw = = \frac{2ab}{\pi c^3} \sum_{k=0}^{\infty} (-1)^k Y_{k+1}(c^2T^2) \frac{\Gamma(k+3/2)}{\Gamma(k+1)} \sum_{m=0}^{k} \binom{k}{m} \frac{(a^2/c^2)^m}{2m+1} \cdot \frac{(b^2/c^2)^{k-m}}{2(k-m)+1}$$

where

(2.12.14) $Y_k(x) = \Gamma(k+1/2, x)/\Gamma(k+1/2)$.

Then

(2.12.15) $$Y_{k+1} = Y_k + \frac{e^{-x} x^{k+1/2}}{\Gamma(k+3/2)} = Y_k + T_k, \qquad k = 0,1,\ldots$$

where $T_{k+1} = T_k \cdot \dfrac{x}{(k+3/2)}$ with $T_o = \dfrac{e^{-x}\sqrt{x}}{(\sqrt{\pi}/2)}$ and $Y_o = \operatorname{erfc}(\sqrt{x})$

Because of the way in which the terms interact, the condition

(2.12.16) $$\frac{a^2+b^2}{c^2} \le \frac{1}{2p} \quad \text{with} \quad p = \begin{cases} 1 & \text{for} \quad c^2T^2 \le 5 \\ c^2T^2 - 4 & \text{for} \quad c^2T^2 > 5 \end{cases}$$

was determined experimentally to give accurate results. Notice also that the binomial coefficients can be generated very rapidly and efficiently by additions using the Pascal triangle relation
$B_{k+1,m} = B_{k,m} + B_{k,m-1}$, $m = 1,k$.

Quadrature for $\mathbf{I_3(a,b,c,T)}$

For the quadrature, we follow the procedure of Chapter 3, Folder 3 by replacing the erf functions with 1 when the arguments exceed 6. As noted, the error is uniformly $O(10^{-15})$ for $\operatorname{erf}(x) = 1$ when $x \ge 6$. Then, we have the same cases as in Folder 3:

Let $W_m = \min\left(\dfrac{6}{a}, \dfrac{6}{b}\right)$, $W_M = \max\left(\dfrac{6}{a}, \dfrac{6}{b}\right)$, $X = \min(a,b)$, and $T = \dfrac{1}{\sqrt{t}}$:

Case I: $1/\sqrt{t} = T \le W_m$

(2.12.17) $$I_3(a,b,c,T) = \int_T^{W_m} e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw + J_5(c,X,W_m) + I_5(c,X,W_M)$$

Case II: $W_m < T \le W_M$

(2.12.18) $$I_3(a,b,c,T) = J_5(c,X,T) + I_5(c,X,W_M)$$

Case III: $W_M < T < \infty$

(2.12.19) $$I_3(a,b,c,T) = \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(cT)$$

where $J_5(a,b,x)$ and $I_5(a,b,x)$ are the I functions of Chapter3, Folder5.

Special Cases

$c = 0$ (Chapter 3, Folder 9 for $\int_0^T \operatorname{erf}(aw)\operatorname{erf}(bw)dw$)

(2.12.20) $$J_3(a,b,0,T) = \int_T^{\infty} \operatorname{erfc}(aw)\operatorname{erfc}(bw)dw$$

$$= \frac{\operatorname{erfc}(aT)\operatorname{ierfc}(bT)}{b} + \frac{e^{-a^2T^2}\operatorname{erfc}(bT)}{a\sqrt{\pi}} - \frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\operatorname{erfc}(T\sqrt{a^2+b^2})$$

a=0 and c=0

(2.12.21) $$J_3(0,b,0,T) = \int_T^{\infty} \operatorname{erfc}(bw)dw = \frac{\operatorname{ierfc}(bT)}{b}$$

c=0 and T=0

(2.12.22) $$J_3(a,b,0,0) = \int_0^{\infty} \operatorname{erfc}(aw)\operatorname{erfc}(bw)dw = \frac{(2/\sqrt{\pi})}{a+b+\sqrt{a^2+b^2}}$$

a=0, c=0, T=0

(2.12.23) $$J_3(0,b,0,0) = \frac{1}{b\sqrt{\pi}} \text{ since } i\operatorname{erfc}(0) = 1/\sqrt{\pi}$$

Inequalities

We derived the relation

(2.12.24) $$W \equiv \frac{\sqrt{\pi}}{2}\operatorname{erfc}(aT)\operatorname{erfc}(bT)\operatorname{erfc}(cT) = aJ_3(b,c,a,T) + bJ_3(a,c,b,T) + cJ_3(a,b,c,T)$$

Following the derivation in the APPENDIX of Chapter 3, Folder 5, divide both sides by $a+b+c$ and we get a convex linear combination of the J_3's. Then

(2.12.25) $\min[J_3(b,c,a,T), J_3(a,c,b,T), J_3(a,b,c,T)] \le \dfrac{W}{a+b+c} \le \max[J_3(b,c,a,T), J_3(a,c,b,T), J_3(a,b,c,T)]$

Also, if we divide by $\sqrt{a^2+b^2+c^2}$ we have the dot product of two vectors on the right, one of which has length 1. By the Cauchy Inequality, we get

(2.12.26) $$\left(\frac{W}{\sqrt{a^2+b^2+c^2}}\right)^2 \le J_3^2(b,c,a,T)+J_3^2(a,c,b,T)+J_3^2(a,b,c,T)$$

Notice that if $c = b = a$, then the max and min are the same and $J_3(a,a,a,T) = \dfrac{W}{3a} = \dfrac{\sqrt{\pi}}{6a}\operatorname{erfc}^3(aT)$ as it should be: $\int_T^\infty e^{-a^2w^2}\operatorname{erfc}^2(aw)dw = \dfrac{\sqrt{\pi}}{6a}\operatorname{erfc}^3(aT)$ since $-\dfrac{2ae^{-a^2w^2}}{\sqrt{\pi}}dw$ is the differential of erfc(aw).

Numerical Considerations

In Chapter 3, Folder 7f, leading terms of each expression are combined to reduce losses of significance by small differences of large numbers. These are implemented in the computer subroutine INTEGI3.

Computer Subroutine

$\mathbf{I_3(a,b,c,T)}$ **or** $\mathbf{I_3^c(a,b,c,T)}$ **: SUBROUTINE INTEGI3(…)** with KODE=1 or KODE=2

References: Chapter 3, Folder 7, Folder 28, Folder 29

(2.13) Functions $\mathbf{I_4(a,b,c,T), I_4^c(a,b,c,T), J_4(a,b,c,T)}$ and $\mathbf{J_4^c(a,b,c,T)}$

$$I_4(a,b,c,T) = \int_T^\infty w^2 e^{-c^2w^2}\,\mathrm{erf}(aw)\,\mathrm{erf}(bw)\,dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^{5/2}}\mathrm{erf}\left(\frac{a}{\sqrt{u}}\right)\mathrm{erf}\left(\frac{b}{\sqrt{u}}\right)du\,,$$

$$I_4^c(a,b,c,T) = \int_T^\infty w^2 e^{-c^2w^2}\,\mathrm{erfc}(aw)\,\mathrm{erfc}(bw)\,dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^{5/2}}\mathrm{erfc}\left(\frac{a}{\sqrt{u}}\right)\mathrm{erfc}\left(\frac{b}{\sqrt{u}}\right)du$$

$$J_4(a,b,c,T) = \int_T^\infty w\, e^{-c^2w^2}\,\mathrm{erf}(aw)\,\mathrm{erf}(bw)\,dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^2}\mathrm{erf}\left(\frac{a}{\sqrt{u}}\right)\mathrm{erf}\left(\frac{b}{\sqrt{u}}\right)du$$

$$J_4^c(a,b,c,T) = \int_T^\infty w\, e^{-c^2w^2}\,\mathrm{erfc}(aw)\,\mathrm{erfc}(bw)\,dw = \frac{1}{2}\int_0^t \frac{e^{-c^2/u}}{u^2}\mathrm{erfc}\left(\frac{a}{\sqrt{u}}\right)\mathrm{erfc}\left(\frac{b}{\sqrt{u}}\right)du$$

$$a \ge 0,\quad b \ge 0,\quad c>0,\quad T = \frac{1}{\sqrt{t}},\quad t > 0$$

Series Representations

For $I_4(a,b,c,T)$ and $I_4(a,b,c,T)$ we have

(2.13.1) $$I_4(a,b,c,T) = \frac{Te^{-c^2T^2}\,\mathrm{erf}(aT)\,\mathrm{erf}(bT)}{2c^2} + \frac{1}{2c^2\sqrt{\pi}}\left\{\frac{ae^{-(a^2+c^2)T^2}\,\mathrm{erf}(bT)}{(a^2+c^2)} + \frac{be^{-(b^2+c^2)T^2}\,\mathrm{erf}(aT)}{(b^2+c^2)}\right\}$$
$$+\frac{ab}{2c^2\sqrt{\pi}}\left\{\frac{1}{a^2+c^2}+\frac{1}{b^2+c^2}\right\}\frac{\mathrm{erfc}(TX)}{X} + \frac{1}{2c^2}I_3(a,b,c,T)$$

(2.13.2) $$I_4^c(a,b,c,T) = \frac{Te^{-c^2T^2}\,erfc(aT)erfc(bT)}{2c^2} - \frac{1}{2c^2\sqrt{\pi}}\left\{\frac{ae^{-(a^2+c^2)T^2}\,erfc(bT)}{(a^2+c^2)} + \frac{be^{-(b^2+c^2)T^2}\,erfc(aT)}{(b^2+c^2)}\right\}$$
$$+\frac{ab}{2c^2\sqrt{\pi}}\left\{\frac{1}{a^2+c^2}+\frac{1}{b^2+c^2}\right\}\frac{\mathrm{erfc}(TX)}{X} + \frac{1}{2c^2}I_3^c(a,b,c,T)$$

where $X^2 = a^2+b^2+c^2$ and $I_3(a,b,c,T)$ and $I_3^c(a,b,c,T)$ are the functions of **(2.12)** and Chapter 3, Folder 7. For $b \to \infty$ and b=0,

(2.13.3) $$I_4(a,\infty,c,T) = \int_T^\infty w^2e^{-c^2w^2}erf(aw)dw = \frac{1}{2c^2}\left[Te^{-c^2T^2}erf(aT) + \frac{a}{a^2+c^2}\frac{e^{-(a^2+c^2)T^2}}{\sqrt{\pi}} + J_5(c,a,T)\right]$$

(2.13.4) $$I_4^c(a,0,c,T) = \int_T^\infty w^2e^{-c^2w^2}erfc(aw)dw = \frac{1}{2c^2}\left[Te^{-c^2T^2}erfc(aT) - \frac{a}{a^2+c^2}\frac{e^{-(a^2+c^2)T^2}}{\sqrt{\pi}} + I_5(c,a,T)\right]$$

where the J_5 and I_5 functions are the functions of **(2.3)** which are computed in Chapter 3, Folder5. Using the symmetry in a and b we can relate these functions by

(2.13.5) $$I_4(a,b,c,T) = \int_T^\infty w^2e^{-c^2w^2}dw - I_4^c(a,0,c,T) - I_4^c(0,b,c,T) + I_4^c(a,b,c,T)$$

(2.13.6) $$I_4^c(a,b,c,T) = \int_T^\infty w^2e^{-c^2w^2}dw - I_4(a,\infty,c,T) - I_4(\infty,b,c,T) + I_4(a,b,c,T)$$

where

$$\int_T^\infty w^2e^{-c^2w^2}dw = \frac{1}{2c^2}\left[Te^{-c^2T^2} + \frac{\sqrt{\pi}}{2c}erfc(cT)\right].$$

For $J_4(a,b,c,T)$ and $J_4^c(a,b,c,T)$ we have

(2.13.7) $$J_4(a,b,c,T) = \frac{e^{-c^2T^2}}{2c^2}\operatorname{erf}(aT)\operatorname{erf}(bT) + \frac{1}{c^2\sqrt{\pi}}\left[bJ_5(\sqrt{b^2+c^2},a,T) + aJ_5(\sqrt{a^2+c^2},b,T)\right]$$

(2.13.8) $$J_4^c(a,b,c,T) = \frac{e^{-c^2T^2}}{2c^2}erfc(aT)erfc(bT) - \frac{1}{c^2\sqrt{\pi}}\left[bI_5(\sqrt{b^2+c^2},a,T) + aI_5(\sqrt{a^2+c^2},b,T)\right]$$

where $J_5(a,b,T)$ and $I_5(a,b,T)$ are the functions of **(2.3)** and Chapter 3, Folder 5. For $b \to \infty$ and b=0,

(2.13.9) $$J_4(a,\infty,c,T) = \frac{1}{2c^2}\left[e^{-c^2T^2}erf(aT) + \frac{a}{\sqrt{a^2+c^2}}erfc(T\sqrt{a^2+c^2})\right]$$

(2.13.10) $$J_4^c(a,0,c,T) = \frac{1}{2c^2}\left[e^{-c^2T^2}erfc(aT) - \frac{a}{\sqrt{a^2+c^2}}erfc(T\sqrt{a^2+c^2})\right]$$

and again using the symmetry in a and b we have the relations

(2.13.11) $$J_4(a,b,c,T) = \frac{1}{2c^2}e^{-c^2T^2} - J_4^c(a,0,c,T) - J_4^c(0,b,c,T) + J_4^c(a,b,c,T)$$

(2.13.12) $$J_4^c(a,b,c,T) = \frac{1}{2c^2}e^{-c^2T^2} - J_4(a,\infty,c,T) - J_4(\infty,b,c,T) + J_4(a,b,c,T)\ .$$

Quadrature for $\mathbf{I_4(a,b,c,T)}$

The outline for this procedure is set in Chapter 3, Folder3 where we replace erf(x) function with 1 when $x \ge 6$ according to the estimate erf(x)=1 + $O(10^{-16})$ and integrate analytically to estimate the tail when c is small. Then the procedure is:

Let $W_m = \min\left(\frac{6}{a},\frac{6}{b}\right)$, $W_M = \max\left(\frac{6}{a},\frac{6}{b}\right)$ $X = \min(a,b)$, $T = \frac{1}{\sqrt{t}}$

Case I: $T \le W_m$

(2.13.13) $$I_4(a,b,c,T) = \int_T^{W_m} w^2 e^{-c^2w^2}\operatorname{erf}(aw)\operatorname{erf}(bw)dw + P(c,X,W_m,W_M)$$

Case II: $W_m < T \le W_M$

(2.13.14) $$I_4(a,b,c,T) = P(c,X,T,W_M)$$

Case III: $W_M < T < \infty$

(2.13.15) $$I_4(a,b,c,T) = \frac{1}{2c^2}\left[Te^{-c^2T^2} + \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(cT)\right]$$

where

(2.13.16) $$P(c,X,L,U) = \frac{1}{2c^2}\left\{Le^{-c^2L^2}\operatorname{erf}(XL) + Ue^{-c^2U^2}\operatorname{erfc}(XU) + \frac{\sqrt{\pi}}{2c}\operatorname{erfc}(cL)\right.$$

$$\left. + \frac{(X/\sqrt{\pi})}{X^2+c^2}\left[e^{-(X^2+c^2)L^2} - e^{-(X^2+c^2)U^2}\right] - \left[I_5(c,X,L) - I_5(c,X,U)\right]\right\}$$

and I_5 is the integral I in Folder 5.

<u>Quadrature for $\mathbf{J_4(a,b,c,T)}$</u>

The outline for this procedure is set in Chapter 3, Folder 3 where we replace erf(x) function with 1 when $x \geq 6$ according to the estimate erf(x)=1 + $O(10^{-16})$. Then the procedure is:

Let $\quad W_m = \min\left(\frac{6}{a}, \frac{6}{b}\right), \quad W_M = \max\left(\frac{6}{a}, \frac{6}{b}\right) \quad X = \min(a,b), \quad T = \frac{1}{\sqrt{t}}$

Then,

<u>Case I</u>, $\quad T \leq W_m$

(2.13.17) $$J_4(a,b,c,T) = \int_T^{W_m} we^{-c^2w^2} \operatorname{erf}(aw)\operatorname{erf}(bw)dw + S(c,X,W_m,W_M) + \frac{1}{2c^2}e^{-c^2W_M^2}$$

<u>Case II</u>, $\quad W_m < T \leq W_M$

(2.13.18) $$J_4(a,b,c,T) = S(c,X,T,W_M) + \frac{1}{2c^2}e^{-c^2W_M^2}$$

<u>Case III</u>, $\quad W_M < T < \infty$

(2.13.19) $$J_4(a,b,c,T) = \frac{1}{2c^2}e^{-c^2T^2}$$

where

(2.13.20) $$\frac{1}{2c^2}e^{-c^2W_M^2} + S(c,X,L,W_M) = \frac{1}{2c^2}\left[e^{-c^2L^2} erf(XL) + e^{-c^2W_M^2} erfc(XW_M)\right] + \frac{X}{2c^2\sqrt{c^2+X^2}}\left[erfc(L\sqrt{c^2+X^2}) - erfc(W_M\sqrt{c^2+X^2})\right]$$

<u>Computer Subroutines</u>

$\mathbf{I_4(a,b,c,T)}$: **SUBROUTINE I4QUAD(…)**
SUBROUTINE I4SER(…)
$\mathbf{J_4(a,b,c,T)}$: **SUBROUTINE J4QUAD(…)**
SUBROUTINE J4SER(…)

References: Chapter 3, Folder 8

(2.14) Functions $\mathbf{I_{14}(a,b,c,T)}$ and $\mathbf{I^c_{14}(a,b,c,T)}$

$$\mathbf{I_{14}(a,b,c,T) = \int_T^\infty e^{-c^2w^2}\frac{erf(aw)erf(bw)}{w^2}dw = \frac{1}{2}\int_0^t e^{-c^2/u}\frac{erf(a/\sqrt{u})erf(b/\sqrt{u})}{\sqrt{u}}du,}$$

$$\mathbf{I^c_{14}(a,b,c,T) = \int_T^\infty e^{-c^2w^2}\frac{erfc(aw)erfc(bw)}{w^2}dw = \frac{1}{2}\int_0^t e^{-c^2/u}\frac{erfc(a/\sqrt{u})erfc(b/\sqrt{u})}{\sqrt{u}}du}$$

$$\mathbf{a>0,\quad b>0,\quad \boldsymbol{T}=\frac{1}{\sqrt{\boldsymbol{t}}}\quad t>0}$$

Series Representation

(2.14.1) $$I_{14}(a,b,c,T) = \frac{I_2(a,b,T)\,e^{-c^2T^2}}{T^2} +$$

$$-2c^2\left[\mathrm{I}_3(\mathrm{a,b,c,T}) + \frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{P}(\sqrt{\mathrm{a}^2+\mathrm{c}^2},\mathrm{b,T}) + \frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{P}(\sqrt{\mathrm{b}^2+\mathrm{c}^2},\mathrm{a,T}) - \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{ab}\sqrt{\pi}}\mathrm{P}(\mathrm{c},\sqrt{\mathrm{a}^2+\mathrm{b}^2},\mathrm{T})\right]$$

$$-2\left[\frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{I}_{13}(\sqrt{\mathrm{a}^2+\mathrm{c}^2},\mathrm{b,T}) + \frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{I}_{13}(\sqrt{\mathrm{b}^2+\mathrm{c}^2},\mathrm{a,T}) - \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{ab}\sqrt{\pi}}\mathrm{I}_{13}(\mathrm{c},\sqrt{\mathrm{a}^2+\mathrm{b}^2},\mathrm{T})\right]$$

where from Chapter3, Folder 9

(2.14.2) $$I_2(a,b,T) = T\,\mathrm{erf}(aT)\,\mathrm{erf}(bT) + \frac{e^{-a^2T^2}}{a\sqrt{\pi}}\mathrm{erf}(bT) + \frac{e^{-b^2T^2}}{b\sqrt{\pi}}\mathrm{erf}(aT) - \frac{\sqrt{a^2+b^2}}{ab\sqrt{\pi}}\mathrm{erf}(T\sqrt{a^2+b^2}),$$

$I_{13}(a,b,T)$ is computed in Chapter 3, Folder 13, $P(a,b,T)$ is computed in Chapter 3, Folder 11, and $I_3(a,b,c,T)$ is computed in Chapter 3, Folder 7. Similarly for $I^c_{14}(a,b,c,T)$,

(2.14.3) $$I^c_{14}(a,b,c,T) = -\frac{I^c_2(a,b,T)\,e^{-c^2T^2}}{T^2} +$$

$$+2c^2\left[-\mathrm{I}^c_3(\mathrm{a,b,c,T}) + \frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{P}^c(\sqrt{\mathrm{a}^2+\mathrm{c}^2},\mathrm{b,T}) + \frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{P}^c(\sqrt{\mathrm{b}^2+\mathrm{c}^2},\mathrm{a,T}) - \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{ab}\sqrt{\pi}}\mathrm{P}^c(\mathrm{c},\sqrt{\mathrm{a}^2+\mathrm{b}^2},\mathrm{T})\right]$$

$$+2\left[\frac{1}{\mathrm{a}\sqrt{\pi}}\mathrm{I}^c_{13}(\sqrt{\mathrm{a}^2+\mathrm{c}^2},\mathrm{b,T}) + \frac{1}{\mathrm{b}\sqrt{\pi}}\mathrm{I}^c_{13}(\sqrt{\mathrm{b}^2+\mathrm{c}^2},\mathrm{a,T}) - \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{ab}\sqrt{\pi}}\mathrm{I}^c_{13}(\mathrm{c},\sqrt{\mathrm{a}^2+\mathrm{b}^2},\mathrm{T})\right]$$

where from Chapter 3, Folder 9

(2.14.4) $$I^c_2(a,b,w) = \text{-werfc(aw)erfc(bw)} + \frac{\mathrm{e}^{-a^2w^2}}{\mathrm{a}\sqrt{\pi}}\mathrm{erfc(bw)} + \frac{\mathrm{e}^{-b^2w^2}}{\mathrm{b}\sqrt{\pi}}\mathrm{erfc(aw)} - \frac{\sqrt{\mathrm{a}^2+\mathrm{b}^2}}{\mathrm{ab}\sqrt{\pi}}\mathrm{erfc}(\mathrm{w}\sqrt{\mathrm{a}^2+\mathrm{b}^2})$$

$$= \frac{\mathrm{T}}{(\mathrm{aT})(\mathrm{bT})}\left[\frac{\mathrm{ierfc}(\mathrm{T}\sqrt{\mathrm{a}^2+\mathrm{b}^2})}{\sqrt{\pi}} - \mathrm{ierfc(aT)ierfc(bT)}\right]$$

and the other complementary functions are treated in the Folders cited above for $I_{14}(a,b,c,T)$.
Using the relation erfc(x)=1-erf(x) we also have

(2.14.5) $$I^c_{14}(a,b,c,T) = \frac{\sqrt{\pi}}{T}ierfc(cT) - I_1(c,a,T) - I_1(c,b,T) + I_{14}(a,b,c,T)$$

(2.14.6) $$I_{14}(a,b,c,T) = \frac{\sqrt{\pi}}{T}\, ierfc(cT) - I_1^c(c,a,T) - I_1^c(c,b,T) + I_{14}^c(a,b,c,T)$$

where $I_1(c,a,T)$, $I_1(c,b,T)$, $I_1^c(c,a,T)$ *and* $I_1^c(c,b,T)$ are computed in Chapter 3, Folder10 and displayed in Section **(2.6).**

<u>Computer Subroutines</u>

$\mathbf{I_{14}(a,b,c,T)}$ **or** $\mathbf{I_{14}^c(a,b,c,T)}$**: SUBROUTINE INTEGI14(…)** with KODE=1 or KODE=2

<u>References</u> Chapter 3, Folders14 and 10

(2.15) Functions $\mathbf{U(a,b,t), V(a,b,t), I_{21}(a,b,c,t), I_{21}^c(a,b,c,t), J_{21}(a,b,c,t), G_n(a,b,T)}$

$$\mathbf{U(a,b,t) = e^{a^2t+2ab}erfc(a\sqrt{t}+b/\sqrt{t})}$$

$$\mathbf{V(a,b,t) = \int_0^t U(a,b,\tau)d\tau,\quad I_{21}(a,b,c,t) = \int_0^t U(a,b,\tau)erf(c/\sqrt{\tau})d\tau}$$

$$\mathbf{I_{21}^c(a,b,c,t) = \int_0^t U(a,b,\tau)erfc(c/\sqrt{\tau})d\tau \qquad J_{21}(a,b,c,t) = \int_0^t \frac{U(a,b,\tau)e^{-c^2/\tau}}{\tau^{3/2}}d\tau}$$

$$\boldsymbol{a>0,\quad b>0,\quad c>0,\quad t>0}$$

$$\mathbf{G_n(a,b,T) = \int_T^\infty \frac{e^{-a^2w^2}i^n erfc(bw)}{w^n}dw,}\quad \boldsymbol{a>0,\quad b\ge 0,\quad T>0,\quad n\ge 0}$$

Representations

(2.15.1) $$V(a,b,t) = \frac{2}{a}\sqrt{\frac{t}{\pi}}e^{-b^2/t} - \left(\frac{1}{a^2}+\frac{2b}{a}\right)\mathrm{erfc}\left(\frac{b}{\sqrt{t}}\right)+\frac{U(t)}{a^2}$$

$$V(a,b,t) = \frac{2\sqrt{t}}{a}ierfc\left(\frac{b}{\sqrt{t}}\right)+\frac{1}{a^2}\left[U(a,b,t)-erfc\left(\frac{b}{\sqrt{t}}\right)\right]$$

(2.15.2) $$I_{21}(a,b,c,t) = V(t)\,\mathrm{erf}(c/\sqrt{t})+\frac{c}{\sqrt{\pi}}\left[\frac{2}{a\sqrt{\pi}}E_1(X)-2\left(\frac{1}{a^2}+\frac{2b}{a}\right)I_5(c,b,T)+\frac{2\sqrt{t}}{a^2\sqrt{\pi}}S_1(a,b,c,t)\right]$$

(2.15.3) $$I_{21}^c(a,b,c,t) = V(t)\,\mathrm{erfc}(c/\sqrt{t})-\frac{c}{\sqrt{\pi}}\left[\frac{2}{a\sqrt{\pi}}E_1(X)-2\left(\frac{1}{a^2}+\frac{2b}{a}\right)I_5(c,b,T)+\frac{2\sqrt{t}}{a^2\sqrt{\pi}}S_1(a,b,c,t)\right]$$

where $X = (b^2+c^2)/t$, I_5 is the I function of Folder 5 and $S_1(a,b,c,t)$ in computational form suitable for a quadrature is

(2.15.4) $$S_1(a,b,c,t) = e^{-X}\int_0^\infty \frac{e^{-2Bw-w^2}}{c^2+(b+w\sqrt{t})^2}dw, \qquad B = a\sqrt{t}+b/\sqrt{t}\,.$$

For J_{21}, we get

(2.15.5) $$J_{21}(a,b,c,t) = \frac{2\sqrt{t}}{\sqrt{\pi}}S_1(a,b,c,t)\,.$$

A series for large parameter L ($L \ge 2$) is

$$X = (b^2+c^2)/t, \qquad B = a\sqrt{t}+b/\sqrt{t}\,,$$

(2.15.6) $$S_1(a,b,c,t) = \frac{e^{-X}}{2tB}\sum_{k=0}^\infty \frac{U_k(x)}{L^k}[e^{B^2}E_{(k+3)/2}(B^2)],$$

$$L = \frac{B}{\sqrt{a^2t+c^2/t}},\quad x = \frac{a\sqrt{t}}{\sqrt{a^2t+c^2/t}}\,,$$

and $U_k(x)$, $k \geq 0$, are Chebyshev polynomials of the second kind which can be generated by forward recurrence on their three-term recurrence relation.
The expressions for $V(t)$, I_{21} and I_{21}^c above contain reciprocal powers of a, but the integrals are analytic functions of a. Therefore, to avoid losses of significance by small differences of large numbers when a is small, we develop the power series in the parameter a. The results are:

(2.15.7) $$U(t) = e^{a^2t+2ab}\,\mathrm{erfc}(a\sqrt{t} + b/\sqrt{t}) = \sum_{n=0}^{\infty} i^n\,\mathrm{erfc}\left(\frac{b}{\sqrt{t}}\right)(-2a\sqrt{t})^n$$

(2.15.8) $$V(t) = 4t\sum_{n=2}^{\infty} i^n\,\mathrm{erfc}\left(\frac{b}{\sqrt{t}}\right)(-2a\sqrt{t})^{n-2}$$

(2.15.9) $$I_{21}(a,b,c,t) = V(t)\,\mathrm{erf}\left(\frac{c}{\sqrt{t}}\right) + \frac{4c\sqrt{t}}{\sqrt{\pi}}e^{-X}\sum_{n=2}^{\infty}(-2a\sqrt{t})^{n-2}y_n\,, \qquad 0 \leq a\sqrt{t} \leq 1$$

(2.15.10) $$I_{21}^c(a,b,c,t) = V(t)\,\mathrm{erfc}\left(\frac{c}{\sqrt{t}}\right) - \frac{4c\sqrt{t}}{\sqrt{\pi}}e^{-X}\sum_{n=2}^{\infty}(-2a\sqrt{t})^{n-2}y_n$$

(2.15.11) $$J_{21}(a,b,c,t) = \frac{e^{-X}}{\sqrt{t}}\sum_{n=0}^{\infty}(-2a\sqrt{t})^n y_n \qquad X = (b^2+c^2)/t,$$

where

(2.15.12) $$y_n = 2T^{n-1}e^X G_n(c,b,T), \qquad , \; G_n(c,b,T) = \int_T^{\infty} e^{-c^2w^2}\frac{i^n\,\mathrm{erfc}(bw)}{w^n}dw \qquad T = \frac{1}{\sqrt{t}}.$$

An extensive stability analysis on the 3-term recurrence for y_n shows that the recurrence must be started with two (quadrature) values near the index [2X].

Recurrence for $\mathbf{y_n = 2T^{n-1}e^X G_n(c,b,T)}$

y_n is scaled to avoid extremes in values from exponentials:

(2.15.13) $$y_n = 2T^{n-1}e^X G_n(c,b,T), \qquad X = (b^2+c^2)/t = (b^2+c^2)T^2$$

(2.15.14) $$n(n+1)y_{n+1} + 2n(bT)y_n + Xy_{n-1} = e^{b^2T^2}i^{n-1}\,\mathrm{erfc}(bT)$$

(2.15.15) $$y_0 = \frac{2e^X}{T}\int_T^{\infty} e^{-c^2w^2}\,\mathrm{erfc}(bw) = \frac{2e^X}{T}I_5(c,b,T)\,, \quad X = (b^2+c^2)/t, \quad T = \frac{1}{\sqrt{t}},$$

(2.15.16) $$y_1 = -bTy_0 + \frac{1}{\sqrt{\pi}}[e^X E_1(X)],$$

Here $I_5(c,b,T)$ is the I function of Folder 5. A stability analysis indcates that the recurrence must be carried out by recurring away from the index N=[2X] to keep homogeneous solutions of the difference equation from amplifying rounding errors.

Special Cases

$c = 0$

(2.15.17) $I_{21}(a,b,0,t) = 0,\qquad I^c_{21}(a,b,0,t) = V(t)$

$a = 0$ for $I_{21}(0,b,c,t)$ Using erf(x)=1-erfc(x), we have

(2.15.18) $I_{21}(0,b,c,t) = 2J^c_3(b,T) - 2W^c_3(b,c,T) = \dfrac{erfc(bT)}{T^2} - \dfrac{2b}{T} ierfc(bT) - 2W^c_3(b,c,T)$.

We chose this form because the computational results are much better than the results in terms of J_3 and W_3. Then, using the formula for W^c_3 from Chapter 3, Folder 10

$$I_{21}(0,b,c,t) = \frac{\mathrm{erf}(cT)\,\mathrm{erfc}(bT)}{T^2} - \frac{2b}{T} ierfc(bT) + \frac{2}{\sqrt{\pi}}\left[bI^c_1(b,c,T) + I^c_1(c,b,T)\right]$$

Combining the iterated coerror functions using the formula from Folder 10,

$$I^c_1(b,c,T) = \frac{\sqrt{\pi}}{T} erfc(cT)ierfc(bT) - \frac{c}{\sqrt{\pi}} E_1(X) + 2bcI_5(c,b,T),$$

improves the computation somewhat by minimizing losses of significance

(2.15.19) $I_{21}(0,b,c,t) = \dfrac{4}{T^2} i^2 erfc(bT) erf(cT) + \dfrac{2c}{\sqrt{\pi}}\left[-\dfrac{b}{\sqrt{\pi}} E_1(X) + 2b^2 I_5(c,b,T) + I^c_1(c,b,T)\right]$

The series for $0 \le a\sqrt{t} \le 1$ with a=0 gives

(2.15.20) $I_{21}(0,b,c,t) = 4ti^2\,\mathrm{erfc}\left(\dfrac{b}{\sqrt{t}}\right)\mathrm{erf}\left(\dfrac{c}{\sqrt{t}}\right) + \dfrac{4c\sqrt{t}}{\sqrt{\pi}} e^{-X} y_2(c,b,T),\qquad \mathrm{T} = 1/\sqrt{\mathrm{t}}$.

$a = 0$ for $I^c_{21}(0,b,c,t)$

For $I^c_{21}(0,b,c,t)$, we have

(2.15.21) $I^c_{21}(0,b,c,t) = \displaystyle\int_0^t \mathrm{erfc}\left(\frac{b}{\sqrt{\tau}}\right)\mathrm{erfc}\left(\frac{c}{\sqrt{\tau}}\right)d\tau = 2W^c_3(b,c,T),\quad \mathrm{T} = 1/\sqrt{\mathrm{t}}$

where W^c_3 is computed in Chapter 3, Folder 10b. From above, we also have

(2.15.22) $I^c_{21}(0,b,c,t) = 4ti^2\,\mathrm{erfc}\left(\dfrac{b}{\sqrt{t}}\right)\mathrm{erfc}\left(\dfrac{c}{\sqrt{t}}\right) - \dfrac{4c\sqrt{t}e^{-X}}{\sqrt{\pi}} y_2(c,b,T),\qquad \mathrm{T} = 1/\sqrt{\mathrm{t}}$.

$a = 0$ for $J_{21}(0,b,c,t)$

(2.15.23) $J_{21}(0,b,c,t) = \displaystyle\int_0^t \mathrm{erfc}\left(\frac{b}{\sqrt{\tau}}\right)\frac{e^{-c^2/\tau}}{\tau^{3/2}} d\tau,\qquad \tau = \frac{1}{w^2}$

$$= 2\int_T^\infty e^{-c^2w^2}\,\mathrm{erfc}(bw)dw = 2I_5(c,b,T),\qquad T = \frac{1}{\sqrt{t}}.$$

$\underline{a = 0,\ c = 0 \text{ for }}\ J_{21}(0,b,0,t)$

(2.15.24) $$J_{21}(0,b,0,t) = 2I_5(0,b,T) = 2\int_T^\infty \operatorname{erfc}(bw)dw = \frac{2}{b} i\operatorname{erfc}(bT)$$

$\underline{a = 0,\ b = 0 \text{ for }}\ I_{21}(0,0,c,t)$

(2.15.25) $$I_{21}(0,0,c,t) = \int_0^t \operatorname{erf}\left(\frac{c}{\sqrt{\tau}}\right)d\tau \qquad \text{(see case for } a = 0\text{)}$$
$$= \frac{\operatorname{erf}(cT)}{T^2} + \frac{2c}{T} i\operatorname{erfc}(cT)$$

$\underline{a = 0 \text{ for }}\ S_1(0,b,c,t)$ (see also case for $a = 0$ for $J_{21}(0,b,c,t)$)

(2.15.26) $$S_1(0,b,c,t) = \frac{\sqrt{\pi}}{\sqrt{t}} I_5(c,b,T), \qquad T = \frac{1}{\sqrt{t}}.$$

$\underline{a = 0,\ b = 0 \text{ for }}\ \mathrm{S}_1(0,0,c,t)$

From above,

(2.15.27) $$S_1(0,0,c,t) = \frac{\sqrt{\pi}}{\sqrt{t}} I_5(c,0,T) = \sqrt{\frac{\pi}{t}}\int_T^\infty e^{-c^2w^2}dw = \frac{\pi}{2c}T\operatorname{erfc}(cT), \qquad T = \frac{1}{\sqrt{t}}$$

$\underline{\text{b=0 for }}\ \mathrm{I}_{21}(a,0,c,t)$

(2.15.28) $$I_{21}(a,0,c,t) = V(a,0,t)\operatorname{erf}\left(\frac{c}{\sqrt{t}}\right) + \frac{c\sqrt{t}}{\sqrt{\pi}}\sum_{n=0}^\infty \frac{(-a\sqrt{t})^n}{\Gamma(\frac{n}{2}+2)} E_{\frac{n+3}{2}}(\frac{c^2}{t})$$

Notice also that

$$V(a,0,t) = 2\ \mathrm{I}_{23}(a\sqrt{t})/a^2 \quad , a \neq 0$$

(2.15.29) $$\mathrm{I}_{23}(x) = (e^{x^2} erfc(x) - 1)/2 + x/\sqrt{\pi} = x^2\sum_{n=0}^\infty \frac{(-x)^n}{\Gamma(\frac{n}{2}+1)(n+2)} = \frac{x^2}{2}\sum_{n=0}^\infty \frac{(-x)^n}{\Gamma(\frac{n}{2}+2)}$$

where I_{23} and related functions are found in Folder 23.

Computer Subroutines

V(a,b,t): DOUBLE PRECISION FUNCTION DVOFT(…)
$\mathbf{I_{21}(a,b,c,t)}$ **or** $\mathbf{I^c_{21}(a,b,c,t)}$ **: SUBROUTINE INTEGI21(…)** with KODE=1 or KODE=2
$\mathbf{J^c_{21}(a,b,c,t)}$ **: SUBROUTINE INTEGJ21(…)**
$\mathbf{S_1(a,b,c,t)}$ **: SUBROUTINE INTEGS1(…)**
$\mathbf{y_n = 2T^{n-1}e^X G_n(c,b,T)}$ **: SUBROUTINE GNSEQ(…)**

References: Chapter 3, Folder 21

(2.16) Functions $\mathbf{I_{22}(a,b,c,t)}$ and $\mathbf{J_{22}(a,b,c,t)}$

$$\mathbf{I_{22}(a,b,c,t)=\int_0^t U(a,b,\tau)\frac{e^{-c^2/\tau}}{\sqrt{\tau}}d\tau}$$

$$\mathbf{J_{22}(a,b,c,t)=\int_0^t U(a,b,\tau)\sqrt{\tau}\, e^{-c^2/\tau}\ d\tau}$$

$$\mathbf{U(a,b,t)=e^{a^2t+2ab}erfc(a\sqrt{t}+b/\sqrt{t})}$$

$$\boldsymbol{a\geq 0,\quad b\geq 0,\quad c\geq 0,\quad t>0}$$

Representations

(2.16.1) $$I_{22}(a,b,c,t)=\frac{1}{a\sqrt{\pi}}E_1(X)-\frac{2\sqrt{t}}{a\sqrt{\pi}}S_2(a,b,c,t)$$

(2.16.2) $$J_{22}(a,b,c,t)=\frac{t}{a\sqrt{\pi}}E_2(X)-\frac{1}{a^2\sqrt{\pi}}\left(b+\frac{1}{2a}\right)E_1(X)+\frac{\sqrt{t}}{a^2}e^{-X}[e^{B^2}\ \mathrm{erfc}(B)]$$

$$-\frac{2c^2\sqrt{t}}{a^2\sqrt{\pi}}S_1(a,b,c,t)+\frac{\sqrt{t}}{a^3\sqrt{\pi}}S_2(a,b,c,t)$$

where $S_1(a,b,c,t)$ was defined in Chapter 3, Folder 21,

(2.16.3) $$S_1(a,b,c,t)=e^{-X}\int_0^\infty \frac{e^{-2Bw-w^2}}{c^2+(b+w\sqrt{t})^2}dw,$$ $B=a\sqrt{t}+b/\sqrt{t}$

$X=(b^2+c^2)/t$

and

(2.16.4) $$S_2(a,b,c,t)=e^{-X}\int_0^\infty \frac{e^{-2Bw-w^2}(b+w\sqrt{t})}{c^2+(b+w\sqrt{t})^2}dw.$$

We also have for $L=B/\sqrt{a^2t+c^2/t}\geq 2$,

(2.16.5) $$S_1(a,b,c,t)=\frac{e^{-X}}{2Bt}\sum_{k=0}^{\infty}\frac{U_k(x)}{L^k}[e^{B^2}E_{(k+3)/2}(B^2)],$$ $B=a\sqrt{t}+b/\sqrt{t}$

$X=(b^2+c^2)/t$

(2.16.6) $$S_2(a,b,c,t)=\frac{e^{-X}}{2\sqrt{t}}\sum_{k=0}^{\infty}\frac{T_k(x)}{L^k}[e^{B^2}E_{(k+2)/2}(B^2)],$$ $x=\dfrac{a\sqrt{t}}{\sqrt{a^2t+c^2/t}}$

where $T_k(x)$ and $U_k(x)$ are Chebyshev polynomials of the first and second kinds. Both polynomials can be generated by forward recurrence on their three-term recurrence relations. While each series converges for $L>1$, we apply the series for $L\geq 2$ to obtain rapid convergence in a numerical evaluation. If $L\geq 2$ is not satisfied, then $S_1(a,b,c,t)$ and $S_2(a,b,c,t)$ are computed by quadrature according to the procedure in the APPENDIX of Chapter 3, Folder 21.

The forms developed above have numerical problems for small a because of small differences of large numbers (an indeterminant form for $a \to 0$). Therefore we develop the power series for small a, $(a\sqrt{t} \le 1)$,

(2.16.7) $$I_{22}(a,b,c,t) = 2e^{-X}\left[b\sum_{n=0}^{\infty} y_n(-2a\sqrt{t})^n + \sqrt{t}\sum_{n=0}^{\infty} (n+2)y_{n+2}(-2a\sqrt{t})^n \right]$$

(2.16.8) $$J_{22}(a,b,c,t) = 4\sqrt{t}e^{-X}\left[b^2\sum_{n=0}^{\infty} y_{n+2}(-2a\sqrt{t})^n + b\sqrt{t}\sum_{n=0}^{\infty} (2n+5)y_{n+3}(-2a\sqrt{t})^n \right.$$

$$\left. + t\sum_{n=0}^{\infty} (n+2)(n+4)y_{n+5}(-2a\sqrt{t})^n \right]$$

where

(2.16.9) $$y_n = 2T^{n-1}e^X G_n(c,b,t) = 2T^{n-1}e^X \int_T^{\infty} e^{-c^2w^2}\frac{\mathrm{i}^n\mathrm{erfc}(bw)}{w^n}dw, \qquad T = 1/\sqrt{t},$$

is computed in **(2.15)** or Chapter 3, Folder 21.

Special Cases

$a = 0$ for

(2.16.10) $$I_{22}(0,b,c,t) = \int_0^t \mathrm{erfc}\left(\frac{b}{\sqrt{\tau}}\right)\frac{e^{-c^2/\tau}}{\sqrt{\tau}}d\tau$$

$$= \frac{1}{T}E_{3/2}(c^2T^2) - 2I_1(c,b,T)$$

where $T = 1/\sqrt{t}$ and I_1 is computed in Folder 10a. We also have

(2.16.11) $$I_{22}(0,b,c,t) = 2\sqrt{t}e^{-c^2/t}\mathrm{erfc}(c/\sqrt{t}) - \frac{2b}{\sqrt{\pi}}E_1(X) - 4c^2I_5(c,b,T)$$

where I_5 is the I function of Folder 5. Notice also that the series for small a at $a = 0$ gives

(2.16.12) $$I_{22}(0,b,c,t) = 2e^{-X}[by_1(c,b,T) + 2\sqrt{t}y_2(c,b,T)]$$

$a = 0$ for

$$J_{22}(0,b,c,t) = \int_0^t \mathrm{erfc}\left(\frac{b}{\sqrt{\tau}}\right)\sqrt{\tau}e^{-c^2/\tau}d\tau,$$

(2.16.13) $$J_{22}(0,b,c,t) = 2\left[\frac{e^{-c^2T^2}erfc(bT)}{3T^3} - \frac{b}{3T^2\sqrt{\pi}}E_2(X) - \frac{2c^2}{3}I_1^c(c,b,T)\right]$$

where I_1^c is defined and computed in Chapter 3, Folder 10b(also = I_1^c of **(2.6)**),

(2.16.14) $$I_1^c(c,b,T) = \frac{\sqrt{\pi}}{T} i\,\mathrm{erfc}(cT) - I_1(c,b,T)$$

in terms of I_1 of Folder 10a,

(2.16.15) $$I_1(c,b,T) = \frac{e^{-c^2T^2}\,\mathrm{erf}(bT)}{T} + \frac{b}{\sqrt{\pi}} E_1(X) - c\sqrt{\pi}\,\mathrm{erfc}(cT) + 2c^2 I_5(c,b,T)$$

where I_5 is the I function of Chapter 3, Folder 5. Also,

(2.16.16) $$J_{22}(0,b,c,t) = \frac{2}{3}\left[\sqrt{t}(t-2c^2)e^{-c^2/t}\,\mathrm{erfc}(bT) - \frac{b}{T^2\sqrt{\pi}} E_2(X) + \frac{2bc^2}{\sqrt{\pi}} E_1(X) + 4c^4 I_5(c,b,T)\right] \qquad X = (b^2+c^2)/t, \qquad T = \frac{1}{\sqrt{t}}$$

We also have from the small a expression for $a = 0$,

(2.16.17) $$J_{22}(0,b,c,t) = 4\sqrt{t}e^{-X}[b^2 y_2(c,b,T) + 5b\sqrt{t}\,y_3(c,b,T) + 8ty_4(c,b,T)]$$

<u>$a = 0,\ b = 0$ for</u>

(2.16.18) $$I_{22}(0,0,c,t) = \int_0^t \frac{e^{-c^2/\tau}d\tau}{\sqrt{\tau}}, \\ = 2\sqrt{\pi t}\ \mathrm{ierfc}(cT), \qquad T = \frac{1}{\sqrt{t}}$$

<u>$a = 0,\ b = 0$ for</u>

(2.16.19) $$J_{22}(0,0,c,t) = \int_0^t \sqrt{\tau}e^{-c^2/\tau}d\tau, \\ = t^{3/2}E_{5/2}(c^2T^2), \qquad T = \frac{1}{\sqrt{t}}.$$

<u>$a = 0,\ c = 0$ for</u>

(2.16.20) $$I_{22}(0,b,0,t) = \int_0^t \frac{\mathrm{erfc}(b/\sqrt{\tau})}{\sqrt{\tau}}\,d\tau \\ = \frac{2\,\mathrm{erfc}(bT)}{T} - \frac{2b}{\sqrt{\pi}} E_1(b^2T^2), \qquad T = \frac{1}{\sqrt{t}}$$

<u>b=0 for $I_{22}(a,0,c,t)$</u>

(2.16.21) $$I_{22}(a,0,c,t) = \sqrt{t}\sum_{n=0}^{\infty} \frac{(-a\sqrt{t})^n}{\Gamma(\frac{n}{2}+1)} E_{\frac{n+3}{2}}\left(\frac{c^2}{t}\right)$$

Relationship Between $\mathbf{I_{22}(a,b,0,t)}$ and $\mathbf{J_{22}(a,b,0,t)}$

(2.16.22) $$I_{22}(a,b,0,t) = 2U(a,b,t)\sqrt{t} + \frac{2at}{\sqrt{\pi}} E_2\left(\frac{b^2}{t}\right) - \frac{2b}{\sqrt{\pi}} E_1\left(\frac{b^2}{t}\right) - 2a^2 J_{22}(a,b,0,t).$$

Asymptotic Expansion For $\mathbf{I_{22}(a,0,c,t)}$ For Large a

(2.16.23) $$I_{22}(a,0,c,t) \sim \frac{e^{-c^2/t}}{a\sqrt{\pi}} \sum_{k=0}^{N} \frac{(-1)^k (1/2)_k}{(ac)^{2k}} [e^{c^2/t}\Gamma(k,\frac{c^2}{t})] + G_N$$

where $$|G_N| \le \frac{(1/2)_{N+1}}{a\sqrt{\pi}(a^2c^2)^{N+1}} \Gamma(N+1,\frac{c^2}{t})$$

and for k=0 and k=1 we have

$$\Gamma(0,x) = \int_x^\infty \frac{e^{-w}}{w} dw = \int_1^\infty \frac{e^{-xv}}{v} dv = E_1(x) \;, \qquad \Gamma(1,x) = \int_x^\infty e^{-w} dw = e^{-x} \;.$$

Forward recurrence on

$$[e^x\Gamma(k+1,x)] = k[e^x\Gamma(k,x)] + x^k, \qquad k = 1,2... \qquad x = c^2/t$$

is numerically stable.

Computer Subroutines

$\mathbf{I_{22}(a,b,c,t)}$**: SUBROUTINE INTEGI22(…)**
$\mathbf{J_{22}(a,b,c,t)}$**: SUBROUTINE INTEGJ22(…)**
$\mathbf{S_2(a,b,c,t)}$**: SUBROUTINE INTEGS2(…)**

References: Chapter 3, Folder 10, Folder 22

(2.17) Function $\mathbf{I_{24}(a,b,c,t)}$, $\mathbf{I^c_{24}(a,b,c,t)}$ and Related Integrals

$$\mathbf{I_{24}(a,b,c,t) = \int_0^t \tau U(\tau)\,erf\left(\frac{c}{\sqrt{\tau}}\right)d\tau}, \qquad \mathbf{I^c_{24}(a,b,c,t) = \int_0^t \tau U(\tau)\,erfc\left(\frac{c}{\sqrt{\tau}}\right)d\tau}$$

$$\mathbf{J_{24}(a,b,t) = \int_0^t \tau U(\tau)\,d\tau \qquad V_{24}(a,b,t) = \int_0^t V(\tau)d\tau}$$

where

$$\mathbf{U(t) = e^{a^2t+2ab}erfc(a\sqrt{t} + b/\sqrt{t})}$$

$$\mathbf{V(t) = \int_0^t U(\tau)\,d\tau = \frac{2}{a}\sqrt{\frac{t}{\pi}}\,e^{-b^2/t} - \left(\frac{1}{a^2} + \frac{2b}{a}\right)erfc\left(\frac{b}{\sqrt{t}}\right) + \frac{U(t)}{a^2}}$$

$$\boldsymbol{a > 0, \quad b > 0, \quad c > 0, \quad t > 0}$$

Representations

For $I_{24}(a,b,c,t)$ we have

(2.17.1) $$I_{24}(a,b,c,t) = tV(t)\,\mathrm{erf}\left(\frac{c}{\sqrt{t}}\right) + \frac{c}{\sqrt{\pi}}T_0 - [T_1 - T_2 - T_3]$$

where $V(t)$ is computed in Folder 21 and

$$T_0(a,b,c,t) = \frac{2t}{a\sqrt{\pi}}E_2(X) - 2\left(\frac{1}{a} + \frac{2b}{a}\right)I_1^c(c,b,T) + \frac{1}{a^2}I_{22}(a,b,c,t)$$

$$T_1(a,b,c,t) = \frac{4}{3a\sqrt{\pi}}\left[\frac{e^{-b^2T^2}\,\mathrm{erf}(cT)}{T^3} + \frac{c}{T^2\sqrt{\pi}}E_2(X) - 2b^2I_1(b,c,T)\right]$$

$$T_2(a,b,c,t) = 2\left(\frac{1}{a^2} + \frac{2b}{a}\right)[J_3(c,T) - W_3(b,c,T)] = 2\left(\frac{1}{a^2} + \frac{2b}{a}\right)[J_3^c(b,T) - W_3^c(b,c,T)]$$

$$T_3(a,b,c,t) = \frac{1}{a^2}I_{21}(a,b,c,t), \qquad T = \frac{1}{\sqrt{t}}, \qquad X = (b^2 + c^2)T^2$$

with I_1 , J_3, J_3^c, W_3, W_3^c *and* I_{22} computed in Folders 10 and 22 of Chapter 3.

For $I_{24}^c(a,b,c,t)$ we have

(2.17.2) $$I_{24}^c(a,b,c,t) = tV(t)\,\mathrm{erfc}\left(\frac{c}{\sqrt{t}}\right) - \frac{c}{\sqrt{\pi}}T_0^c - [T_1^c - T_2^c + T_3^c]$$

where

$$T_0^c = T_0$$

$$T_1^c = \frac{4}{3a\sqrt{\pi}}\left[\frac{e^{-b^2T^2}\,erfc(cT)}{T^3} - \frac{c}{T^2\sqrt{\pi}}E_2(X) - 2b^2I_1^c(b,c,T)\right]$$

$$T_2^c = 2\left(\frac{1}{a^2}+\frac{2b}{a}\right)W_3^c(b,c,T)$$

$$T_3^c = \frac{1}{a^2}I_{21}^c(a,b,c,t), \qquad T=\frac{1}{\sqrt{t}}, \qquad X=(b^2+c^2)T^2$$

with I_1^c , J_3, J_3^c, W_3, W_3^c *and* I_{22}^c computed in Folders 10 and 22 of Chapter 3.

For $J_{24}(a,b,t)$ we have

(2.17.3) $$J_{24}(a,b,t)=\int_0^t \tau U(\tau)d\tau = \lim_{c\to\infty} I_{24}(a,b,c,t) = tV(t)-\int_0^t V(\tau)d\tau$$

and

(2.17.4) $$J_{24}(a,b,t)=\left(t-\frac{1}{a^2}\right)V(t)-\frac{2}{aT^3\sqrt{\pi}}E_{5/2}(b^2T^2)+\frac{4}{T^2}\left(\frac{1}{a^2}+\frac{2b}{a}\right)i^2\,\mathrm{erfc}(bT)$$

$$J_{24}(a,b,t)=\left(t-\frac{1}{a^2}\right)V(t)-\frac{8}{aT^3}i^3 erfc(bT)+\frac{4}{a^2T^2}i^2\,\mathrm{erfc}(bT)$$

It follows that

(2.17.5) $$V_{24}(a,b,t)=\int_0^t V(\tau)d\tau = \frac{1}{a^2}V(a,b,t)+\frac{2}{aT^3\sqrt{\pi}}E_{5/2}(b^2T^2)-\frac{4}{T^2}\left(\frac{1}{a^2}+\frac{2b}{a}\right)i^2\,\mathrm{erfc}(bT)$$
$$=\frac{1}{a^2}V(a,b,t)+\frac{8}{aT^3}i^3 erfc(bT)-\frac{4}{a^2T^2}i^2\,\mathrm{erfc}(bT)$$

In all of these, computation for small a results in losses of significance by small differences of large numbers or an indeterminant form for $a\to 0$ since each integral is analytic in the parameter a. Resolution of these forms for $a\to 0$ yields the power series in $a\sqrt{t}$ which converges best for $a\sqrt{t}<1$:

(2.17.6) $$I_{24}(a,b,c,t)=J_{24}(a,b,t)\,\mathrm{erf}\left(\frac{c}{\sqrt{t}}\right)+\frac{8cte^{-X}}{\sqrt{\pi}}\left[b\sum_{n=0}^{\infty}(-2a\sqrt{t})^n y_{n+3}+\sqrt{t}\sum_{n=0}^{\infty}(n+2)(-2a\sqrt{t})^n y_{n+4}\right]$$

(2.17.7) $$I_{24}^c(a,b,c,t)=J_{24}(a,b,t)\,\mathrm{erfc}\left(\frac{c}{\sqrt{t}}\right)-\frac{8cte^{-X}}{\sqrt{\pi}}\left[b\sum_{n=0}^{\infty}(-2a\sqrt{t})^n y_{n+3}+\sqrt{t}\sum_{n=0}^{\infty}(n+2)(-2a\sqrt{t})^n y_{n+4}\right]$$

(2.17.8) $$J_{24}(a,b,t) = tV(t)-V_{24}(a,b,t)$$

(2.17.9) $$V_{24}(a,b,t)=16t^2\sum_{n=0}^{\infty}(-2a\sqrt{t})^n i^{n+4}\,\mathrm{erfc}(bT)$$

where $V(t)$ and the $y_n(c,b,T)$ sequence are computed in Chapter 3, Folder 21,

(2.17.10) $$V(t)=4t\sum_{n=0}^{\infty}(-2a\sqrt{t})^n i^{n+2}\,\mathrm{erfc}(bT) \qquad T=\frac{1}{\sqrt{t}}, \qquad X=(b^2+c^2)T^2$$

(2.17.11) $$y_n(c,b,T)=2T^{n-1}e^X\int_T^{\infty} e^{-c^2w^2}\frac{i^n\,\mathrm{erfc}(bw)}{w^n}dw.$$

The iterated coerror functions are generated in subroutine DINERFC and the $y_n(c,b,T)$ sequence is genrerated in subroutine GNSEQ.

Computer Subroutines

$\mathbf{I_{24}(a,b,c,t)}$ or $\mathbf{I^{c}_{24}(a,b,c,t)}$: **SUBROUTINE INTEGI24(…)** with KODE=1 or KODE=2
$\mathbf{J_{24}(a,b,t)}$: **SUBROUTINE INTEGJ24(…)**
$\mathbf{V_{24}(a,b,t)}$: **SUBROUTINE INTEGV24(…)**

References: Chapter 3, Folders 10, 21, 22, 24

(2.18) Function $\mathbf{I_{25}(a,b,c,d,t)}$

$$\mathbf{I_{25}(a,b,c,d,t) = \int_0^t U(a,b,\tau)\,U(c,d,\tau)\,d\tau}$$

$$\mathbf{U(a,b,t) = e^{a^2 t + 2ab}\,erfc(a\sqrt{t} + b/\sqrt{t})}$$

$$\boldsymbol{a \geq 0,\quad b \geq 0,\quad c \geq 0,\quad d \geq 0,\quad t > 0}$$

Representation

(2.18.1) $$I_{25}(a,b,c,d,t) = \frac{1}{(a^2+c^2)}\Bigg\{a^2 U(c,d,t)V(a,b,t) + \frac{c}{\sqrt{\pi}}R_1(a,b,d,t) - \frac{d}{\sqrt{\pi}}R_2(a,b,d,t)$$

$$- \frac{2ac^2}{\sqrt{\pi}}J_{22}(c,d,b,t) + c^2(1+2ab)I_{21}^c(c,d,b,t) + \frac{c}{\sqrt{\pi}}I_{22}(a,b,d,t) - \frac{d}{\sqrt{\pi}}J_{21}(a,b,d,t)\Bigg\}$$

where

$$V(a,b,t) = \frac{2}{a}\sqrt{\frac{t}{\pi}}e^{-b^2/t} - \frac{(1+2ab)}{a^2}\mathrm{erfc}\left(\frac{b}{\sqrt{t}}\right) + \frac{1}{a^2}U(a,b,t)$$
$$= \frac{2\sqrt{t}}{a}ierfc\left(\frac{b}{\sqrt{t}}\right) + \frac{1}{a^2}\left[U(a,b,t) - erfc\left(\frac{b}{\sqrt{t}}\right)\right]$$

$$R_1(a,b,d,t) = \frac{2at}{\sqrt{\pi}}E_2(X) - 2(1+2ab)I_1^c(d,b,T)$$

$$R_2(a,b,d,t) = \frac{2a}{\sqrt{\pi}}E_1(X) - 2(1+2ab)I_5(d,b,T)$$

$$T = \frac{1}{\sqrt{t}}, \qquad\qquad X = (b^2+d^2)/t.$$

Here, I_5 and I_1^c are computed in Folders 5 and 10. V, J_{21} I_{21}^c, I_{22} and J_{22} are the principal results of Folders 21 and 22. Notice also that I_{25} is symmetric in the pairs (a,b) and (c,d) and exchanging these pairs on the right yields an alternate form. In fact, any convex linear combination of these forms gives I_{25}; in particular, adding and dividing by 2 gives the symmetric form.

Computer Subroutine

$\mathbf{I_{25}(a,b,c,d,t)}$: SUBROUTINE INTEGI25(…)

Reference Chapter 3, Folder 25

(2.19) Function $I_{26}(a,b,c,d,t)$

$$I_{26}(a,b,c,d,t) = \int_0^t \tau U(a,b,\tau)U(c,d,\tau)d\tau$$

$$U(a,b,t) = e^{a^2t+2ab}\,\mathrm{erfc}(a\sqrt{t} + b/\sqrt{t})$$

$$a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 0, \quad t > 0$$

Series Representations

Symmetric form

(2.19.1)

$$\begin{aligned} I_{26}(a,b,c,d,t) = \frac{1}{(a^2+c^2)}\{ & tU(a,b,t)U(c,d,t) - I_{25}(a,b,c,d,t) \\ & + \frac{a}{\sqrt{\pi}} J_{22}(c,d,b,t) - \frac{b}{\sqrt{\pi}} I_{22}(c,d,b,t) \\ & + \frac{c}{\sqrt{\pi}} J_{22}(a,b,d,t) - \frac{d}{\sqrt{\pi}} I_{22}(a,b,d,t)\} \end{aligned}$$

where I_{22}, J_{22} and I_{25} are defined in Chapter 3, Folders 22 and 25. A more efficient computational form is also presented in terms of more fundamental integrals which eliminates redundant computation.

An alternate, non-symmetric, more complicated form is also derived in Chapter 3, Folder 26.

Computer Subroutine

$I_{26}(a,b,c,d,t)$: SUBROUTINE INTEGI26(…)

Reference: Chapter 3, Folder 26

(2.20) Indefinite Integrals

$$\mathbf{J = \int \; e^{(a^2-b^2)x} erfc\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right)dx}$$

$$\mathbf{I = \int \; xe^{(a^2-b^2)x} erfc\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right)dx}$$

Representations

(2.20.1) $$J \equiv \int \; e^{(a^2-b^2)x} \operatorname{erfc}\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right)dx$$

$$= \frac{e^{(a^2-b^2)x}}{a^2-b^2} \operatorname{erfc}\left(a\sqrt{x} + \frac{c}{\sqrt{x}}\right) + \frac{e^{-2ac}}{2}\left[-f(\sqrt{x},a,b,c) + f(\sqrt{x},-a,b,-c)\right]$$

where $f(\sqrt{x},a,b,c) = \dfrac{e^{2bc}}{(a^2-b^2)}\left(1+\dfrac{a}{b}\right)\operatorname{erfc}\left(b\sqrt{x}+\dfrac{c}{\sqrt{x}}\right)$

(2.20.2) $$I = -\frac{e^{(a^2-b^2)x}}{a^2-b^2}\left[-x + \frac{1}{a^2-b^2}\right]\operatorname{erfc}[X(\sqrt{x},a,c)]$$

$$+\frac{e^{-2ac}}{2}[F(\sqrt{x},a,b,c) - F(\sqrt{x},-a,b,-c)]$$

$$F(\sqrt{x},a,b,c) = \frac{e^{2bc}}{a^2-b^2}\left\{\frac{-\sqrt{x}}{b\sqrt{\pi}} e^{-X^2(\sqrt{x},b,c)}\left(1+\frac{a}{b}\right)\right.$$

$$\left.+\left[\left(\frac{c}{b}+\frac{1}{a^2-b^2}\right)\left(1+\frac{a}{b}\right)-\frac{a}{2b^3}\right]\operatorname{erfc}[X(\sqrt{x},b,c]\right\}$$

$$X(\sqrt{x},a,c) = a\sqrt{x} + c/\sqrt{x}$$

Computer Subroutines

I: SUBROUTINE ERFINT(…)

References: Chapter 3, Folder 12

(2.21) Functions $\mathbf{H_{23}(x)}$, $\mathbf{I_{23}(x)}$ and $\mathbf{J_{23}(x)}$

$$\mathbf{H_{23}(x) = \int_0^x e^{w^2} erfc(w)\, dw = \frac{1}{2}\int_0^{x^2} e^{v} erfc(\sqrt{v})\frac{dv}{\sqrt{v}}}$$

$$\mathbf{I_{23}(x) = \int_0^x e^{w^2} erfc(w)\, w\, dw = \frac{1}{2}\int_0^{x^2} e^{v} erfc(\sqrt{v})\, dv}$$

$$\mathbf{J_{23}(x) = \int_0^x e^{w^2} erfc(w)\, w^2 dw = \frac{1}{2}\int_0^{x^2} e^{v} erfc(\sqrt{v})\sqrt{v}\, dv}$$

$$\boldsymbol{x \geq 0}$$

Representations

H_{23} is evaluated from the relations

(2.21.1)
$$H_{23}(x) = \begin{cases} S(x,0) & 0 \leq x \leq 1 \\ S_2 + S(x,2) & 1 < x \leq 3 \\ S_4 + S(x,4) & 3 < x \leq 4 \\ S_4 + V(x,4) & 4 < x < \infty \end{cases}$$

where $S(x,x_0)$ is the Taylor expansion about x_0,

(2.21.2)
$$S(x,x_0) = \sum_{n=0}^{\infty} (-2)^n [e^{x_0^2} i^n \operatorname{erfc}(x_0)] \frac{(x-x_0)^{n+1}}{n+1},$$

(2.21.3)
$$S_2 = \int_0^2 e^{v^2} \operatorname{erfc}(v)dv, \qquad S_4 = \int_0^4 e^{v^2} \operatorname{erfc}(v)dv,$$

(2.21.4) $S_2 = 0.9753620874841564$ $\qquad$ $S_4 = 1.344468257503159.$

and $V(x,4)$ is the integral

(2.21.5)
$$V(x,4) = \int_4^x y(v)dv \qquad y(v) = e^{v^2} \operatorname{erfc}(v) = \frac{1}{v\sqrt{\pi}} \sideset{}{'}\sum_{n=0}^{\infty} a_{2r} T_{2r}\left(\frac{4}{v}\right), \qquad v \geq 4,$$

where $y(v)$ is a Chebyshev sum with coefficients a_{2r} (see Chapter 3, Folder 23 for numerical values). The result for $V(x,4)$ is

(2.21.6)
$$V(x,4) = \frac{1}{\sqrt{\pi}}\left[\sideset{}{'}\sum_{r=0}^{\infty} a_{2r} A_{2r}(x) + \ln\left(\frac{x}{4}\right)\right]$$

where $A_{2r}(x)$ is computed from the recurrence

$$A_0(x) = 0, \qquad A_2(x) = 1 - \left(\frac{4}{x}\right)^2$$

(2.21.7)
$$A_{2r+2}(x) + A_{2r}(x) = -\frac{1}{2r(r+1)} - \left[\frac{T_{2r+2}(4/x)}{2r+2} - \frac{T_{2r}(4/x)}{2r}\right], \qquad r \geq 1.$$

Truncation of $V(x,4)$ at 18 terms suffices for errors $O(10^{-16})$. Here, the prime on the sum means to halve the first term.

$I_{23}(x)$ can be evaluated explicitly,

(2.21.8) $$I_{23}(x) = \frac{e^{x^2}\,\mathrm{erfc}(x) - 1}{2} + \frac{x}{\sqrt{\pi}}$$

and $J_{23}(x)$ can be expressed in terms of $H_{23}(x)$,

(2.21.9) $$J_{23}(x) = \frac{1}{2}\left[xe^{x^2}\,\mathrm{erfc}(x) + \frac{x^2}{\sqrt{\pi}} - H_{23}(x) \right]$$

Reduction Formula

Let

(2.21.10) $$I_\alpha(x) = \int_0^x e^{w^2}\,\mathrm{erfc}(w) w^\alpha\, dw, \qquad \alpha = 2n \text{ or } 2n{+}1, \quad \mathrm{n} = 1,2,\ldots.$$

Then

(2.21.11) $$\mathrm{I}_\alpha(\mathrm{x}) = \frac{1}{2}\mathrm{x}^{\alpha-1}\mathrm{erfc}(\mathrm{x}) + \frac{1}{\sqrt{\pi}}\int_0^{\mathrm{x}} \mathrm{w}^{\alpha-1}\mathrm{dw} - \frac{(\alpha-1)}{2}\int_0^{\mathrm{x}} \mathrm{e}^{\mathrm{w}^2}\mathrm{erfc}(\mathrm{w})\mathrm{w}^{\alpha-2}\mathrm{dw}$$

(2.21.12) $$\mathrm{I}_\alpha(\mathrm{x}) = \frac{1}{2}\mathrm{x}^{\alpha-1}\mathrm{erfc}(\mathrm{x}) + \frac{1}{\alpha\sqrt{\pi}}\mathrm{x}^\alpha - \frac{(\alpha-1)}{2}\mathrm{I}_{\alpha-2}(\mathrm{x}).$$

Notice that for $\alpha = 2$ we get the formula for $J_{23}(x)(=I_2(x))$ since $I_0(x) = H_{23}(x)$. Since $\alpha = 2n$ or $2n+1$, a repeated application of this formula ends up with $I_0(x)$ or $I_1(x)$ which translate to $H_{23}(x)$ or $I_{23}(x)$ respectively.

Numerically, all of the formulas derived so far suffer from high cancellation of significant digits when x is small which results in a loss of relative error. Consequently the power series is valuable in numerical evaluation.

Power Series

(2.21.13) $$I_\alpha(x) = \int_0^x e^{w^2}\,\mathrm{erfc}(w) w^\alpha\, dw = x^{\alpha+1}\sum_{n=0}^{\infty} \frac{(-x)^n}{\Gamma\left(\frac{n}{2}+1\right)(n+\alpha+1)}, \qquad \alpha = 0, 1, 2, \ldots$$

where $\mathrm{I}_0(\mathrm{x}) = \mathrm{H}_{23}(\mathrm{x}), \mathrm{I}_1(\mathrm{x}) = \mathrm{I}_{23}(\mathrm{x})$ and $\mathrm{I}_2(\mathrm{x}) = \mathrm{J}_{23}(\mathrm{x})$

Asymptotic Expansion for $\mathbf{H_{23}(x)}$ for $\mathbf{x \to \infty}$

(2.21.14) $$H_{23}(x) = \int_0^x e^{w^2} erfc(w)dw = \frac{1}{2}\int_0^{x^2} \frac{e^\tau erfc(\sqrt{\tau})}{\sqrt{\tau}} d\tau \sim \frac{1}{2\sqrt{\pi}}\left[\gamma + 2\ln(2x) - \sum_{n=1}^{\infty}\frac{A_n}{x^{2n}}\right]$$

where γ is the Euler constant

$$\gamma = 0.5772156649015328606 \text{ and } A_n = \frac{(-1)^n (1/2)_n}{n}.$$

Computer Subroutines

$H_{23}(x)$: DOUBLE PRECISION FUNCTION DHERFC(…)

References: Chapter 3, Folder 23

(2.22) Incomplete Bessel Function

$$\boldsymbol{I(a,b,X) = \int_X^\infty e^{-at-b/t} dt},$$
$$\boldsymbol{a > 0,\ \ b > 0,\ \ X > 0}$$

Representations

Small b/X expansion

(2.22.1) $$I = \frac{e^{-aX}}{a} + X\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\left(\frac{b}{X}\right)^k E_k(aX)$$

Small aX expansion

(2.22.2) $$I = 2\sqrt{\frac{b}{a}}K_1(2\sqrt{ab}) - X\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(aX)^k E_{k+2}\left(\frac{b}{X}\right)$$

A representation in terms of analytic functions suitable for a quadrature,

(2.22.3) $$I(a,b,X) = \sqrt{\frac{b}{a}}\int_{\ln(X\sqrt{a/b})}^{\infty} e^{-2\sqrt{ab}\cosh v+v} dv$$

The incomplete Gamma functions are explored in Chaudhry and Zubar where Equation 2.163 expresses I in terms of

(2.22.4) $$\Gamma(\alpha, x : b) = \int_x^\infty t^{\alpha-1}\exp(-t - b/t)dt = \frac{1}{2\pi i}\sum_{n=0}^{\infty} \frac{(-b)^n}{n!}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)x^{\alpha-n-s}}{s+n-\alpha}ds,$$

which for $\alpha = 1$ is $I(1, b, x)$. Evaluation of the residues in the complex integral results in the small b/X expansion. Notice that a change of variables t=v/a in $I(a,b,X)$ produces

(2.22.5) $$I(a,b,X) = \frac{1}{a}I(1,ab,aX) = \frac{1}{a}\Gamma(1,aX : ab)$$

Special Cases

X=0 gives the Bessel Function of order 1

(2.22.6) $$I(a,b,0) = \int_0^\infty e^{-at-b/t}dt = 2\sqrt{\frac{b}{a}}K_1(2\sqrt{ab})$$

a=0 gives

(2.22.7) $$I(0,b,X) = \int_X^\infty e^{-b/t}dt = X\ E_2(b/X).$$

b=0 gives

(2.22.8) $$I(a,0,X) = \int_X^\infty e^{-at}dt = \frac{e^{-aX}}{a}$$

References: Chapter 3, Folder17;

Chaudhry, M. A. and Zubair, S. M. "On A Class of Incomplete Gamma Functions with Applications" Chapman & Hall/CRC, Boca Raton, 2001